Section 2.10

Nuclear charge density distribution from e^- scattering

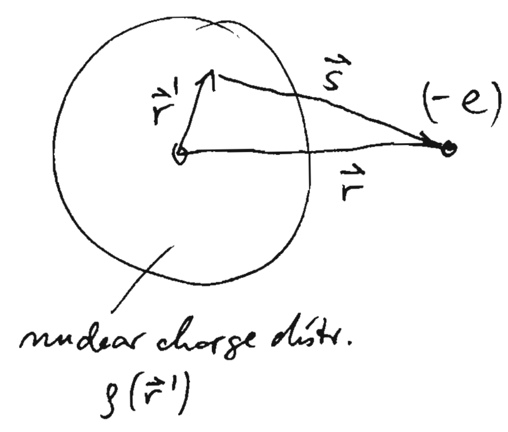
The most precise measurements of the nuclear charge density $\rho_c(\vec{r}\,')$ are made via elastic e^- scattering experiments. To see how one can extract $\rho_c(\vec{r}\,')$ from $d\sigma/d\Omega$, let us quickly review QM scattering theory (e.g. Shankar, QM, chapter 19):

$$\underbrace{\frac{d\sigma(\theta,\varphi)}{d\Omega}}_{\text{diff. cross section}} = |\underbrace{f(\theta,\varphi)}_{\text{scattering amplitude.}}|^2$$

In first-order Born approximation, the scattering amplitude is proportional to the Fourier transform of the scattering potential $V(\vec{r})$ with respect to the momentum transfer $\vec{q} = \vec{k}_f - \vec{k}_i$:

$$f(\theta,\varphi) = -\left(\frac{\mu}{2\pi\hbar^2}\right)\int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, V(\vec{r})$$

where $|\vec{q}\,|^2 = 4k^2\sin^2(\theta/2)$ contains the scatt. angle dependance.



The interaction potential between the electron point charge $(-e)$ and the nuclear charge distribution due to protons is given by

$$V(\vec{r}) = -e\int \frac{\rho_c(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'$$

Inserting $V(\vec{r})$ into the expression for $f(\theta,\varphi)$ one finds

$$f(\theta,\varphi) = +\left(\frac{\mu e}{2\pi\hbar^2}\right)\int d^3r\, e^{-i\vec{q}\cdot\vec{r}} \int \frac{\rho_c(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'$$

Introduce the vector $\vec{s} = \vec{r} - \vec{r}\,'$ (see figure) one obtains (details see K. Heyde, p. 47-48):

$$f(\theta,\varphi) = \underbrace{\left(\frac{\mu e}{2\pi\hbar^2}\right)\int d^3\vec{s}\, \frac{e^{-i\vec{q}\cdot\vec{s}}}{|\vec{s}|}}_{f_{Ruth}(q)} \underbrace{\int d^3\vec{r}\,'\, \rho_c(\vec{r}\,')\, e^{-i\vec{q}\cdot\vec{r}\,'}}_{F(\vec{q})}$$

For a spherical nuclear charge distribution, $F(\vec{q})$ reduces to the 1-D integral

$$\boxed{F(q) \propto \int_0^\infty \rho_c(r')\, \frac{\sin(qr')}{(qr')}\, 4\pi r'^2 dr'}$$

$$\Rightarrow \boxed{\frac{d\sigma(\theta)}{d\Omega} = \left(\frac{d\sigma(\theta)}{d\Omega}_{Ruth}\right)\cdot F^2(q)}$$

The correction factor (compared to 2 point ~~charges~~ charges) is called the charge nuclear form factor $F^2(q)$, with

$q = 2k\sin(\theta/2)$.

Hence, measurements of $d\sigma(\theta)/d\Omega$ as a function of momentum transfer q can be used to extract info about nuclear charge distribution $\rho_c(r')$.

- Discuss slides on e^- scatt. exp. and $\rho_c(r')$ distributions for various nuclei; compared to HF theory.

Nuclear "root mean square" (= rms) radii

a) charge radii

Denoting the proton charge distribution by $\rho_p(\vec{r})$, we define the mean square charge radius via

$$\boxed{\langle r^2\rangle_p = \frac{1}{Z}\int d^3r\, r^2 \rho_p(\vec{r})}$$

If the nucleus is spherical, we can simplify: $\rho_p(\vec{r}) \to \rho_p(r)$ and $\int d^3r \to \int d\Omega \int r^2 dr$

resulting in

$$\langle r^2\rangle_p = \frac{1}{Z}\underbrace{\int d\Omega}_{=4\pi}\int dr\, r^4 \rho_p(r) = \frac{1}{Z}\, 4\pi \int_0^\infty dr\, r^4 \rho_p(r)$$

In the special case $Z=1$, we recover the rms-radius of the proton already discussed in Section 2.0b (slide 3).

The rms-radius is defined by taking the square root:

$$\boxed{(R_{rms})_p = \sqrt{\langle r^2\rangle_p}}$$

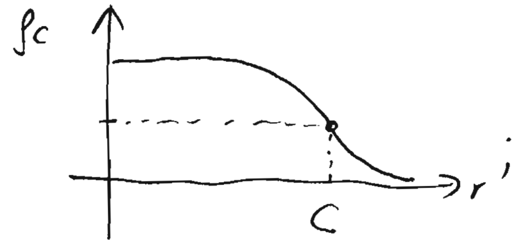
b) neutron and mass radii

Similarly, one defines mean square radii for the neutron distribution and for the mass distribution:

$$\langle r^2\rangle_n = \frac{1}{N}\int d^3r\, r^2 \rho_n(\vec{r})$$

$$\langle r^2\rangle_A = \frac{1}{A}\int d^3r\, r^2 \underbrace{[\rho_n(\vec{r}) + \rho_p(\vec{r})]}_{\equiv \rho(\vec{r})}.$$

One often parametrizes $\rho_c(r')$ by a <u>Fermi distribution</u> of the form



$$\boxed{\rho_c(r') = \frac{\rho_{c0}}{1 + \exp[(r'-C)/a]}}$$

Here <u>C is the half-density radius</u> and the parameter a determines the surface thickness. The normalization integral

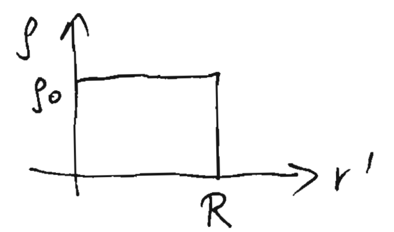
$$\int \rho_c(\vec{r}\,')\, d^3r' = Ze$$

fixes the density parameter ρ_{c0} in terms of the 2 remaining parameters C and a. For nuclei with mass numbers $20 \le A \le 208$ one finds from exp. fits:

$$\boxed{\begin{aligned} C &\approx 1.12 \cdot A^{1/3}\ \text{fm} \\ a &\approx 0.57\ \text{fm} \end{aligned}}$$

<u>Ref:</u> Eisenberg & Greiner, Vol. 2

If we approximate the density further by a uniform distribution of the form



$$\Longrightarrow \quad \boxed{R = r_0 \cdot A^{1/3}\ ; \ r_0 \simeq 1.1\ \text{fm}}$$

and we compute the following value for the <u>constant nuclear density ρ_0</u>:

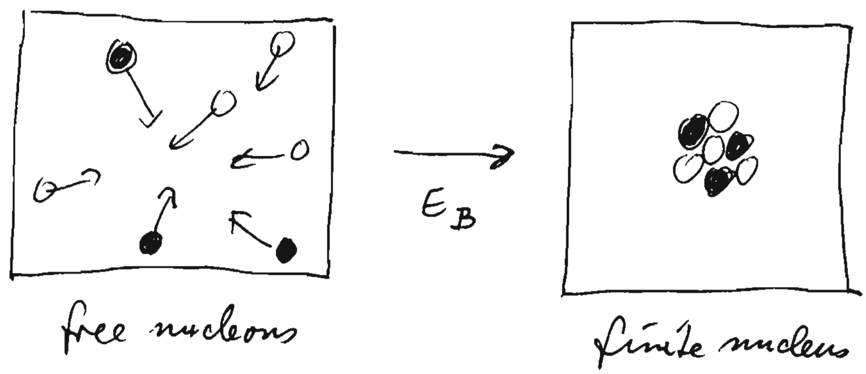
$$\rho_0 = \frac{A}{\frac{4\pi}{3} R^3} = \frac{3A}{4\pi r_0^3 A} = \frac{3}{4\pi r_0^3} = \frac{3}{4\pi (1.1\ \text{fm})^3}$$

$$\boxed{\rho_0 \simeq 0.17\ \text{fm}^{-3}}$$

Nuclear binding energy and semi-empirical mass formula

<u>Ref</u>: Ring & Schuck, chapter 1.1

The binding energy E_B of a nucleus is the energy that is released when free nucleons (Z protons and N neutrons) condense into a bound nucleus:



Because $E = mc^2$, we can express E_B in terms of mass differences:

$$\boxed{E_B\Big|_{exp} = m_{Nucleus}(Z,A)\cdot c^2 - Z\cdot m_p c^2 - N\cdot m_n c^2}$$

<u>Experimentally</u>, one can determine <u>binding energies</u> by measuring the masses of nuclear isotopes in a <u>mass spectrograph</u>. See slides of nuclear chart in Section 2.1.
Experimentally, we know the E_B's for about 3,000 isotopes!

> Challenge for nuclear many-body theories:
> explain E_B and $\rho_{p,n}(\vec{r})$ for all isotopes

Theoretically, we can calculate E_B and $\rho_{p,n}(\vec{r})$ both for known and unknown (n-rich) nuclei within mean field theories (HF, HFB). In particular, we can express

E_B as the expectation value of the many-body nuclear Hamiltonian H within the many-body ground state wave function $|\phi_0\rangle$:

$$\boxed{E_B\Big|_{\text{theory}} = \langle \phi_0 | H | \phi_0 \rangle}$$

More about this in the following section 2.2

One of the earliest attempts to understand the measured binding energies $E_B(Z,N)$ is the "semi-empirical mass formula" developed by Bethe and Weizsäcker (1935–36):

$$\boxed{E_B(Z,N) = E_{vol} + E_{surf} + E_{coul} + E_{sym} + E_{pair}}$$

Z = charge number, N = neutron number
$A = Z + N$ = mass number

nomenclature:
${}^{A}_{Z}$ Chem. symbol $\rightarrow {}^{120}_{50}Sn$

with (see textbook of K. Heyde, p. 219):

Volume energy: $E_{vol} = a_v \cdot A$, $a_v = -15.85$ MeV

Surface energy: $E_{surf} = a_s \cdot A^{2/3}$, $a_s = +18.34$ MeV

Coulomb energy: $E_{coul} = a_c \cdot \dfrac{Z(Z-1)}{A^{1/3}}$, $a_c = 0.71$ MeV

Symmetry energy: $E_{sym} = a_{sym} \dfrac{(N-Z)^2}{A}$, $a_{sym} = 23.21$ MeV

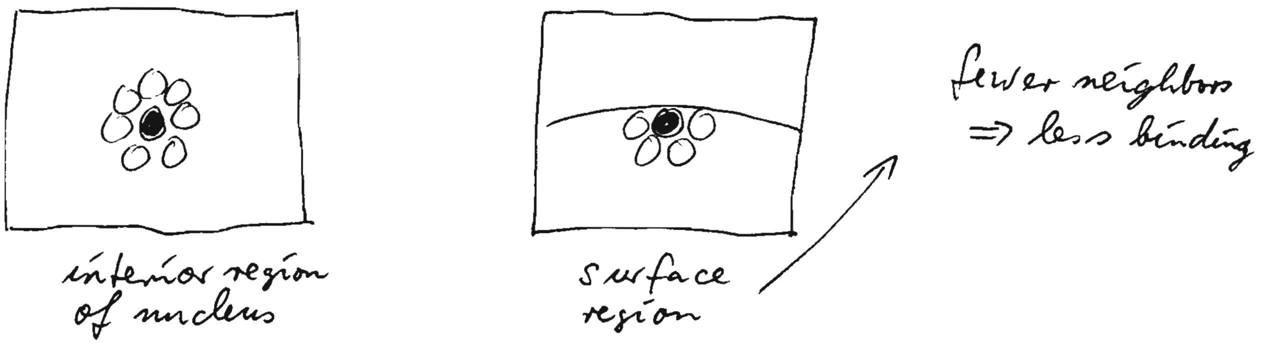
pairing energy:

$$E_{pair} = \begin{cases} -12\,\text{MeV} \cdot A^{-1/2} & \text{for even-even nuclei} \\ 0 & \text{for even-odd} \quad '' \\ +12\,\text{MeV} \cdot A^{-1/2} & \text{for odd-odd} \quad '' \end{cases}$$

The constants $a_v, a_s, a_c, \ldots$ given above were obtained from a fit to measured binding energies by Wapstra (1971).

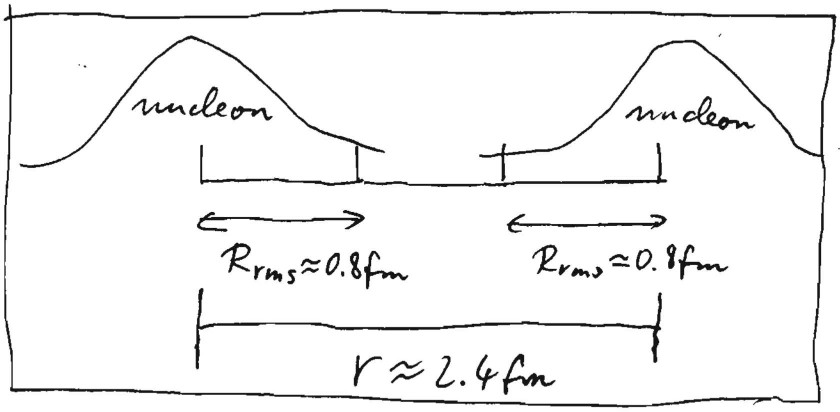
Comments:

- The volume energy is proportional to R^3; using $R = r_0 A^{1/3} \Rightarrow E_{vol} \propto A$
- The surface energy is proportional to $R^2 \Rightarrow E_{surf} \propto A^{2/3}$. Note that E_{surf} has opposite sign to E_{vol}, hence reduces binding!



- Coulomb energy $\propto \frac{Z^2}{R} \propto \frac{Z^2}{A^{1/3}}$.
- Symmetry energy: QM in origin (Pauli principle for nucleons)! Explained by Fermi gas model, see Section 2.3
- Pairing energy: QM in origin, can be explained by "short-range correlations" caused by "residual interaction" → later, Section 3.3

The first 3 terms in $E_B(Z,N)$ suggest that a nucleus might be understood in terms of a charged liquid drop which gives rise to E_{vol}, E_{surf}, and E_{coul} [Bohr & Wheeler, 1939]. However, we know today that nuclei in their ground states (or at moderate $E^* \lesssim 20$ MeV) are not liquids at all; the reason is the Pauli principle which keeps the nucleons on average ~ 2.4 fm apart:



Hence, scattering events are scarce and the mean free path is of order $2R = 2 r_0 \cdot A^{1/3}$. [In a classical liquid, the mean free path is small and there is lot of scattering!]. These basic features can be explained by the "pair correlation function" for fermions, see Section 2.3.

We note in passing that in relativistic h. i. collisions, one may achieve densities $(5-10) * \rho_0$. Under these conditions, a hydrodynamic description becomes much more realistic.

- Discuss slides for this section 2.1

(a) $E_B(Z,N)$ varies for most nuclei between $(-7.5$ to $-8.5)$ MeV per nucleon.

(b) $E_B(Z,A)$ explains energy gain in fission of heavy nuclei and fusion of light nuclei.

(c) From semi-empirical mass formula we can obtain the "valley of stability" → homework!
