2.2 Binding energies and Q-values for nuclear reactions

Consider the following nuclear reaction

$$\underbrace{Z_1, N_1 \; + \; Z_2 N_2 \; + \dots}_{\text{"entrance channel } \underline{\alpha}\text{"}} \longrightarrow \underbrace{Z_1', N_1' \; + \; Z_2' N_2' \; + \dots}_{\text{"reaction channel } \underline{\beta}\text{"}}$$

In this section, we will derive an expression for the <u>reaction Q-value $Q_{\alpha\beta}$</u> using overall energy conservation.

<u>Practical applications:</u>

1) Energy release in spontaneous fission
2) Energy release in fusion reaction
3) "open" and "closed" reaction channels in heavy-ion reactions, calculate threshold energy

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<u>Derivation of $Q_{\alpha\beta}$ from energy conservation</u>

<u>Ref:</u> Blatt & Weisskopf, Theor. Nucl. Physics (p.4, 311-317)

We write the total energy of a nucleus in the form

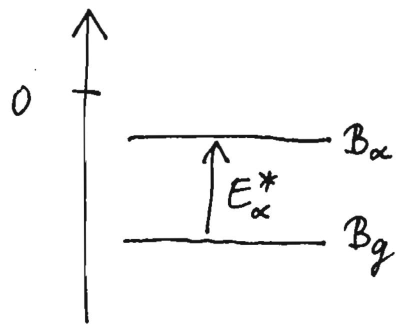
$$E_\alpha^{tot} = \underset{\text{rest energy}}{m_\alpha(Z,N)c^2} + \underset{\text{kin. energy}}{T_\alpha(Z,N)} \qquad \underline{(1)}$$

with

$$m_\alpha(Z,N)\,c^2 = \underbrace{Z \cdot m_p c^2}_{\text{protons}} + \underbrace{N \cdot m_n c^2}_{\text{neutrons}} + \underbrace{B_\alpha(Z,N)}_{\text{binding energy}} \qquad \underline{(2)}$$

where the <u>binding energy is negative ($B_\alpha < 0$).</u>

Note that eq. (2) allows for the possibility that the nuclei may be in an excited state $|\alpha\rangle$ in which case the binding energy $|B|$ is reduced, i.e.



$$B_\alpha = B_g + E_\alpha^* \qquad \underline{(3)}$$

binding energy of excited state (<0)

g.s. binding energy (<0)

excitation energy (>0)

To derive the $Q_{\alpha\beta}$ value, we use overall energy conservation during the reaction, i.e.

$$E_\alpha^{tot}(t \to -\infty) \stackrel{!}{=} E_\beta^{tot}(t \to +\infty) \qquad \underline{(4)}$$

Note that before the reaction $(t \to -\infty)$ ~~day~~ and after the reaction $(t \to +\infty)$ the <u>nuclei/reaction products are far apart</u>, so that there is <u>no interaction</u> between them any more. Hence, inserting (1) into (4) we obtain

$$\left.\begin{aligned} E_\alpha^{tot}(t \to -\infty) &= \sum_{i=1}^{n} \left[T_{\alpha_i}(Z_i, N_i) + m_{\alpha_i}(Z_i, N_i)\,c^2 \right] = \\ = E_\beta^{tot}(t \to +\infty) &= \sum_{k=1}^{m} \left[T_{\beta_k}(Z_k', N_k') + m_{\beta_k}(Z_k', N_k')\,c^2 \right] \end{aligned}\right\} \qquad \underline{(5)}$$

Note that expression (5) is very general. It allows for n nuclei in the entrance channel and for m nuclei in the reaction channel β.

We now define the total kinetic energy T_α, T_β in the channels α and β:

$$T_\alpha = \sum_{i=1}^{n} T_{\alpha_i}(Z_i, N_i) \qquad T_\beta = \sum_{k=1}^{m} T_{\beta_k}(Z_k', N_k') \qquad (6)$$

The Q-value $Q_{\alpha\beta}$ is defined as the total kin. energy difference

$$\boxed{Q_{\alpha\beta} \stackrel{\text{def}}{=} T_\beta - T_\alpha} \qquad (7)$$

Inserting (6) into (5) we find from (7):

$$Q_{\alpha\beta} = T_\beta - T_\alpha = \sum_{i=1}^{n} m_{\alpha_i}(Z_i, N_i)c^2 - \sum_{k=1}^{m} m_{\beta_k}(Z_k', N_k')c^2 \qquad (8)$$

i.e. the Q-value is the "mass" difference between entrance and reaction channels.

We now insert the expression (2) for the masses in terms of binding energies. Assuming that the total number of protons and neutrons does not change in the nuclear reaction (non-rel. collisions!), the terms of the form

$$Z \cdot m_p c^2 \quad \text{and} \quad N \cdot m_n c^2$$

cancel out, and we are left with binding energy differences for the reaction channels α, β:

$$Q_{\alpha\beta} = T_\beta - T_\alpha = \sum_{i=1}^{n} B_{\alpha_i}(Z_i, N_i) - \sum_{k=1}^{m} B_{\beta_k}(Z_k', N_k'). \qquad (9)$$

Insert (3) into (9):

$$Q_{\alpha\beta} = T_\beta - T_\alpha = \sum_{i=1}^{n} B_g(Z_i, N_i) + \sum_{i=1}^{n} E^*_{\alpha_i}(Z_i, N_i) - \sum_{k=1}^{m} B_g(Z'_k, N'_k) - \sum_{k=1}^{m} E^*_{\beta_k}(Z'_k, N'_k) \qquad (10)$$

We now define the ground state to ground state Q-value

$$\boxed{Q_{gg} \stackrel{\text{def}}{=} \sum_{i=1}^{n} B_g(Z_i, N_i) - \sum_{k=1}^{m} B_g(Z'_k, N'_k)} \qquad (11a)$$

Inserting the def. (11a) into (10), we can express $Q_{\alpha\beta}$ in terms of Q_{gg} and excitation energy differences (E^*):

$$\boxed{Q_{\alpha\beta} \stackrel{\text{def}}{=} T_\beta - T_\alpha = Q_{gg} + \sum_{i=1}^{n} E^*_{\alpha_i}(Z_i, N_i) - \sum_{k=1}^{m} E^*_{\beta_k}(Z'_k, N'_k)} \qquad (11b)$$

Our final result is given by equations (11a, b).

Applications of Q-value formalism

1. Energy release in spontaneous symmetric fission of ^{238}U

Here we have one nucleus in the entrance channel ($n=1$) and 2 fission fragments in the reaction channel ($m=2$).
From (11b):

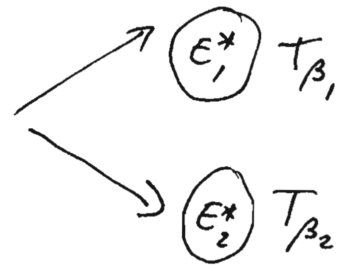
$$Q_{\alpha\beta} = T_\beta - T_\alpha = Q_{gg} + E_\alpha^*(Z,N) - \underbrace{\left[E_{\beta_1}^*\left(\tfrac{Z}{2},\tfrac{N}{2}\right) + E_{\beta_2}^*\left(\tfrac{Z}{2},\tfrac{N}{2}\right)\right]}_{\stackrel{\text{def}}{=} E_\beta^*}.$$

We assume now that the fissioning nucleus is in its ground state ($E_\alpha^* = 0$); this is called spontaneous fission, and that the nucleus is at rest ($T_\alpha = 0$) $\Longrightarrow$

$$Q_{\alpha\beta} = T_\beta = Q_{gg} - E_\beta^*$$

From this, we obtain the total energy released in spontaneous symmetric fission:

$$\boxed{\Delta E = T_\beta + E_\beta^* = Q_{gg}}$$



This expression tells us that

a) the energy is released in the form of kinetic energy of the 2 fission fragments (T_{β_1}, T_{β_2}) and in the form of excitation energies of the fragments.

b) ΔE is equal to the Q-value Q_{gg}

which we are now going to compute. From (11a) we have

$$Q_{gg} = B_g(Z,N) - 2\cdot B_g\left(\tfrac{Z}{2},\tfrac{N}{2}\right).$$

We look up the binding energies [NOTE: $B_g < 0$] at the

Website of the National Nuclear Data Center (NNDC)

at Brookhaven, see Bibliography section link. We find

$$\frac{B_g}{A}(^{238}U) = -7.570116 \text{ MeV} \Rightarrow B_g = -1{,}801.69 \text{ MeV } (^{238}U)$$

$$\frac{B_g}{A}(^{119}Pd) = -8.36896 \text{ MeV} \Rightarrow B_g = -995.91 \text{ MeV } (^{119}Pd)$$

resulting in $Q_{gg} = -1{,}801.69 + 2 \cdot 995.91 \text{ MeV}$

$$\boxed{Q_{gg}(\text{fission}) = +190. \text{ MeV} \mathrel{\hat{=}} 0.79 \text{ MeV/nucleon}}$$

Note that asymmetric fission is far more likely for U; for neutron-induced fission of ^{235}U, e.g.

$$^{235}_{92}U + n \rightarrow {}^{140}_{54}Xe + {}^{94}_{38}Sr + 2n$$

one can easily calculate the corresponding Q-value from our general formalism presented above.

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② Fusion reactor: deuteron + triton fusion

$$^{2}_{1}H + {}^{3}_{1}H \rightarrow {}^{4}_{2}He + n$$

deuteron triton

binding energies from NNDC Website:

$$\frac{B_g}{A}(^{2}_{1}H) = -1.1122 \text{ MeV} \Rightarrow B_g(^{2}_{1}H) = -2.224 \text{ MeV}$$

$$\frac{B_g}{A}\left({}^3_1H\right) = -2.8272\,\text{MeV} \Rightarrow B_g\left({}^3_1H\right) = -8.482\,\text{MeV}$$

$$\frac{B_g}{A}\left({}^4_2He\right) = -7.0739\,\text{MeV} \Rightarrow B_g\left({}^4_2He\right) = -28.295\,\text{MeV}$$

n (neutron): $B_g = 0$

$$\rightarrow Q_{gg} = B_g\left({}^2H\right) + B_g\left({}^3H\right) - B_g\left({}^4He\right) - \underbrace{B_g(n)}_{=0} =$$

$$= -2.224 - 8.482 + 28.295\,\text{MeV}$$

$$\boxed{Q_{gg}\,(\text{fusion}) = +17.59\,\text{MeV} \mathrel{\hat{=}} 3.5\,\text{MeV/nucleon}}$$

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(3) Criterion for "open" and "closed" reaction channels

From the def. of $Q_{\alpha\beta}$, eq. (7), we obtain

$$T_\beta = T_\alpha + Q_{\alpha\beta}$$

reaction channel (T_β)　　entrance channel (T_α)

Obviously, a reaction can only occur if the total kin. energy of the reaction products $T_\beta \geq 0$ $\Rightarrow$

<u>reaction channel</u> <u>β is "open"</u> if <u>$T_\beta = T_\alpha + Q_{\alpha\beta} \geq 0$</u>

" " β is <u>"closed"</u> if <u>$T_\beta = T_\alpha + Q_{\alpha\beta} < 0$</u>

reaction cannot occur

(4) Deep-inelastic heavy-ion reaction, "threshold energy"

$$(Z_1\, N_1) + (Z_2\, N_2) \longrightarrow (Z_1'\, N_1') + (Z_2'\, N_2')$$

projectile target

entrance channel $|\alpha\rangle$

reaction channel $|\beta\rangle$
(assume 2 fragments)

In a typical heavy-ion reaction experiment, projectile and target nuclei are in their ground state at $t \to -\infty$ (i.e. $E^*_{\alpha_1} = 0 = E^*_{\alpha_2}$); we denote this channel by $|\alpha_0\rangle$. From (11b) we find

$$Q_{\alpha_0 \beta} = T_\beta - T_{\alpha_0} = Q_{gg} + 0 - E^*_{\beta_1} - E^*_{\beta_2}$$

Solve for kin. energy in entrance channel, T_{α_0}:

$$\boxed{T_{\alpha_0} = T_\beta - Q_{gg} + E^*_{\beta_1} + E^*_{\beta_2}}$$

The "threshold energy" is the minimum kin. energy in the entrance channel, $T^{min}_{\alpha_0}$, for a reaction to take place. Since $E^* \geq 0$, it is given by the condition $E^*_{\beta_1} = 0 = E^*_{\beta_2}$ (reaction channel $|\beta_0\rangle$), and we find

$$\boxed{T^{min}_{\alpha_0} = T_{\beta_0} - Q_{gg}}$$ "threshold energy"

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(5) Heavy-ion fusion reaction

see homework!