Consider now the spin-part of the N-N wave function. We can couple the two $s=\frac{1}{2}$ fermions antiparallel ($S=0$) or parallel ($S=1$); one obtains in the coupled representation, see e.g. Shankar, QM, p.405:

see e.g. Shankar, QM, p.405

spin-singlet: $$\chi^{(1,2)}_{S=0,S_z=0} = \frac{1}{\sqrt{2}}\left[|\uparrow(1)\rangle|\downarrow(2)\rangle - |\downarrow(1)\rangle|\uparrow(2)\rangle \right]$$

where we have used the shorthand notation $|\uparrow\rangle \equiv |s=\frac{1}{2}, m_s=+\frac{1}{2}\rangle$ etc.

spin-triplet: $$\begin{cases} \chi^{(1,2)}_{S=1,S_z=+1} = |\uparrow(1)\rangle|\uparrow(2)\rangle \\ \chi^{(1,2)}_{S=1,S_z=0} = \frac{1}{\sqrt{2}}\left[|\uparrow(1)\rangle|\downarrow(2)\rangle + |\downarrow(1)\rangle|\uparrow(2)\rangle \right] \\ \chi^{(1,2)}_{S=1,S_z=-1} = |\downarrow(1)\rangle|\downarrow(2)\rangle \end{cases}$$

We define the spin-exchange operator $\hat{P}_\sigma$ via

$$\hat{P}_\sigma\, |S, S_z; (1,2)\rangle = |S, S_z; (2,1)\rangle$$

from which we conclude that

$$\boxed{\begin{array}{ll} \hat{P}_\sigma\, |S=0\rangle = -\,|S=0\rangle & \text{antisym.} \\ \hat{P}_\sigma\, |S=1\rangle = +\,|S=1\rangle & \text{sym.} \end{array}}$$

It is also of interest to calculate the exp. values of spin-dep. operators in the spin-singlet and triplet states. For example, let us look at the ~~[illegible]~~ 3rd term in the v_{18} potential.

$$\langle v_3 \rangle = \langle S, S_z^{(1,2)} |\, (\vec{\sigma}_1 \cdot \vec{\sigma}_2) \,| S, S_z^{(1,2)} \rangle\, v_3(r)$$

From $\vec{\sigma}^2 = (\vec{\sigma}_1 + \vec{\sigma}_2)^2 = \underset{\overset{\parallel}{3}}{\vec{\sigma}_1^{\,2}} + \underset{3}{\vec{\sigma}_2^{\,2}} + 2\vec{\sigma}_1\cdot\vec{\sigma}_2$ we obtain

$$\underline{\vec{\sigma}_1\cdot\vec{\sigma}_2} = \frac{\vec{\sigma}^2 - 6}{2} = \frac{\vec{\sigma}^2}{2} - 3 = \underline{\frac{2}{\hbar^2}\vec{S}^2 - 3} \qquad (*)$$

where we have used the identity $\vec{S} = \frac{\hbar}{2}(\vec{\sigma}_1 + \vec{\sigma}_2) = \frac{\hbar}{2}\vec{\sigma}$.

Inserting $(*)$ into the matrix element, we obtain

$$\langle v_3 \rangle = v_3(r)\, \langle S, S_z \,|\, \tfrac{2}{\hbar^2}\vec{S}^2 - 3 \,|\, S, S_z \rangle$$

Since the eigenvalues of $\vec{S}^2 = \hbar^2 S(S+1)$ we finally obtain:

$$\boxed{\langle v_3 \rangle = v_3(r) \times \begin{cases} (-3) & ; \text{ for } S=0 \\ (+1) & ; \text{ for } S=1 \end{cases}}$$

In a completely analogous fashion, one can construct the corresponding isospin-states for the N-N system; from $\vec{T} = \vec{t}_1 + \vec{t}_2 = \frac{1}{2}(\vec{\tau}_1 + \vec{\tau}_2)$ etc. one finds:

<u>isospin-singlet</u>: $\zeta^{(1,2)}_{T=0,\,T_z=0} = \frac{1}{\sqrt{2}}\left[\,|p(1)\rangle|n(2)\rangle - |n(1)\rangle|p(2)\rangle\right]$

where $|p\rangle$ = proton and $|n\rangle$ = neutron.

<u>isospin-triplet</u>:
$$\begin{cases} \zeta^{(1,2)}_{T=1,\,T_z=+1} = |p(1)\rangle|p(2)\rangle \\ \zeta^{(1,2)}_{T=1,\,T_z=0} = \frac{1}{\sqrt{2}}\left[\,|p(1)\rangle|n(2)\rangle + |n(1)\rangle|p(2)\rangle\right] \\ \zeta^{(1,2)}_{T=1,\,T_z=-1} = |n(1)\rangle|n(2)\rangle \end{cases}$$

In analogous fashion to spin, one defines the <u>isospin-exchange op. $\hat{P}_\tau$</u>

$$\hat{P}_\tau\,|T, T_z; (1,2)\rangle = |T, T_z; (2,1)\rangle$$

from which we obtain

$$\boxed{\begin{array}{ll} \hat{P}_\tau \, |T=0\rangle = -\,|T=0\rangle & \text{antisym.} \\ \hat{P}_\tau \, |T=1\rangle = +\,|T=1\rangle & \text{sym.} \end{array}}$$

By combining space-, spin-, and isospin-parts one obtains the total N-N state vector.

a) Uncoupled representation

$$|\psi(1,2)\rangle_{\text{unc.}} = \underset{\substack{\text{relative}\\ \text{motion}}}{|L, M_L\rangle} \otimes \underset{\text{spin}}{|S, S_z\rangle} \otimes \underset{\text{isospin}}{|T, T_z\rangle}$$

b) Coupled representation

Here we couple the angular momenta L and S to J:

$$\underbrace{|L\,S; J\,M_J\rangle}_{\text{coupled repr.}} = \sum_{M_L, S_z} \underbrace{\langle L\,M_L, S\,S_z | J\,M_J\rangle}_{\text{Clebsch-Gordan}} \underbrace{|L, M_L\rangle |S, S_z\rangle}_{\text{uncoupled repr.}}$$

and the total N-N state vector becomes

$$\boxed{|\psi(1,2)\rangle_{\text{coupled}} = \underset{\substack{\text{rel. motion}\\ \text{+ Spin}}}{|L\,S; J\,M_J\rangle} \otimes \underset{\text{isospin}}{|T, T_z\rangle}}$$

Spin multiplicity

spectroscopic notation of quantum states: $\boxed{{}^{(2S+1)}L_J}$

Total antisymmetry of state vector $|\psi(1,2)\rangle$

Pauli's spin-statistics theorem for fermions requires antisymmetry under coordinate $1 \leftrightarrow 2$ exchange:

$$\hat{P}_{12}\, |\psi(1,2)\rangle \overset{\text{def}}{=} |\psi(2,1)\rangle \overset{!}{=} -\,|\psi(1,2)\rangle$$

Since particle interchange $\hat{P}_{12}$ affects position, spin, and isospin we have

$$\hat{P}_{12} = -1 = \hat{P}_r \cdot \hat{P}_\sigma \cdot \hat{P}_\tau \qquad (*)$$

The operator $\hat{P}_r$ is here the same as the parity operator because $\vec{r} = \vec{r}_1 - \vec{r}_2 \rightarrow -\vec{r}$ under $(1,2)$ interchange.

From $(*)$ we conclude for the eigenvalues of $\hat{P}_r, \hat{P}_\sigma, \hat{P}_\tau$:

$$\boxed{P_\tau = -P_r \cdot P_\sigma \quad \text{with } P_r = \text{parity} = (-1)^L}$$

On page 9 we have shown that

$$P_\sigma = \begin{cases} -1 & \text{for } S=0 \text{ (spin-singlet)} \\ +1 & \text{for } S=1 \text{ (spin-triplet)} \end{cases}$$

Since P_r and P_σ determine, via the Pauli principle, the allowed P_τ-value, they determine the allowed pp, np, nn-combinations in these states!

From p. 10, 11 we observe

$$P_\tau = \begin{cases} -1 & \text{for } T=0 \text{ (isospin-singlet): only } np \text{ possible!} \\ +1 & \text{for } T=1 \text{ (isospin-triplet): } pp, nn, \text{ and } np. \end{cases}$$

The table on p. 13 shows how the Pauli principle restricts the allowed states of the N-N system. We use the "spectroscopic notation"

Spin multiplicity

$$\boxed{{}^{(2S+1)}L_J}$$

orbital total

ang. momentum

with the nomenclature

$S \hat{=} (L=0)$

$P \hat{=} (L=1)$

$D \hat{=} (L=2)$ etc.

Low-L states of the nucleon-nucleon system

L	S	J	Spectroscopic notation	parity $(-1)^L = P_r$	P_σ	$P_\tau \overset{\text{Pauli}}{\equiv} -P_r \cdot P_\sigma$	T	allowed N-N combinations
0	0	0	${}^1S_0$	+1	−1	+1	1	pp, nn, np
	1	1	${}^3S_1$	+1	+1	−1	0	np
1	0	1	${}^1P_1$	−1	−1	−1	0	np
	1	0	${}^3P_0$	−1	+1	+1	1	pp, nn, np
		1	${}^3P_1$	−1	+1	+1	1	pp, nn, np
		2	${}^3P_2$	−1	+1	+1	1	pp, nn, np
2								

student homework!

Exp. data on pp and np elastic scattering

Diff. cross sections have been measured in a wide energy range:

$$\boxed{\text{slide } \#7 : \quad \frac{d\sigma(\theta_{cm})}{d\Omega_{cm}} \text{ for } E_{lab} = 10-500 \text{ MeV}}$$

Recall theory of elastic scattering (no spin);
partial wave expansion method (e.g. Shankar, QM, p. 545-550)

diff. scatt. cross section $\frac{d\sigma(\theta_{cm})}{d\Omega_{cm}} = |f(\theta_{cm})|^2$

with the scattering amplitude

$$f(\theta_{cm}) = \frac{1}{k} \sum_{L=0}^{\infty} (2L+1)\, e^{i\delta(L)} \sin\delta(L)\, P_L(\cos\theta_{cm})$$

where $E_{cm} = \hbar^2 k^2/2\mu$, and the phase shifts $\delta(L)$ are determined by solving the radial Schrödinger eq. for a given potential $V(r)$. From the (numerical) soln. of the radial WF's one obtains the phase shifts $\delta(L)$:

$$R_L(r \to \infty) = A_L \frac{\sin\left(kr - L\frac{\pi}{2} + \delta(L)\right)}{r}$$

These phase shifts are energy-dependent (E_{cm}/E_{lab}).

Conclusion: From a "phase shift analysis" of elastic scatt. data one can get info about $V(r)$.

Phase shift analysis of Argonne v-18 potential

Because nucleons carry spin, the partial wave analysis is more complicated. In particular, the phase shifts depend on

$$\underset{\text{no spin}}{\delta(L)} \longrightarrow \underset{\text{for N-N scatt}}{\delta(L,S,J)} \equiv \delta(\underbrace{^{2S+1}L_J}_{\text{spectroscopic notation.}})$$

Again, the phase shifts depend on the beam energy E_{lab}.

Discuss slide 12: phase shifts $\delta(^1S_0)$ and
slide 14: $\delta(^3P_0)$
slide 16: { phase shifts for 12 "reaction channels" $^{2S+1}L_J$.

The phase shift analysis requires that one calculate the expectation values of $V_{NN} = \sum_{k=1}^{18} v_k(r) O_k$ ← spin/isospin/LS...

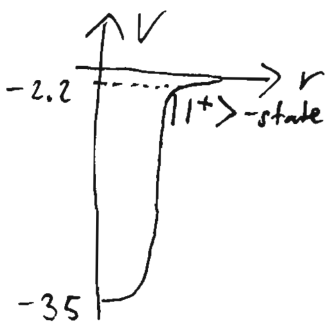
in the various reaction channels:

$$V_{NN}(^{2S+1}L_J) = \langle ^{2S+1}L_J | V_{NN} | ^{2S+1}L_J \rangle$$

These quantities are shown for the (older) Reid soft-core potential in slide 9. We note:

slide 9: a) N-N interaction has short-range repulsive core ($V_{NN} \to \infty$ for $r = 0.5-1.0$ fm)
b) long-range character determined by OPEP
↝ strong spatial-spin-isospin correlations.

Exp. evidence for tensor force (S_{12}): the deuteron



Experiments reveal that there is only one (weakly) bound state for 2 nucleons, namely the deuteron (pn) system. The binding energy is only −2.2 MeV, and the bound state carries the quantum numbers $J^\pi = 1^+$.

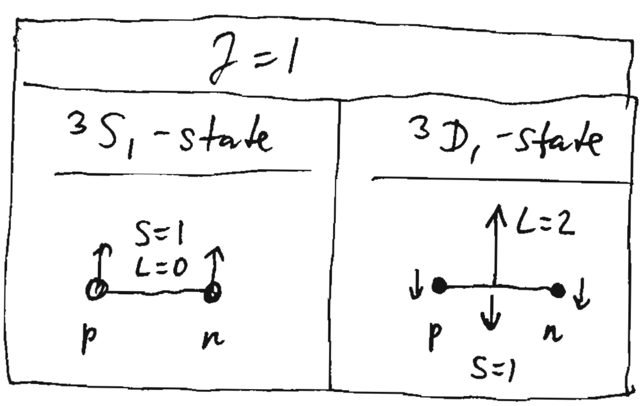
Our table of the NN quantum states, p. 13, reveals one possible state with the correct quantum numbers, namely the

$$|{}^3S_1\rangle\text{-state} \quad (L=0,\ S=1,\ J=1).$$

However, experiments reveal that the deuteron ground state has a nonzero quadrupole moment $Q = 2.82$ mb; since an $L=0$ state is spherically symmetric, there must be an admixture with $L>0$. A fit to exp. data reveals that a mixed state of the form

$$|\Psi^{g.s.}_{deuteron}\rangle = a\,|{}^3S_1\rangle + b\,|{}^3D_1\rangle$$

with $|b|^2 = 0.03 \pm 0.01$ describes all the features.



Note: since L is not conserved here (superposition of $L=0,2$) the V_{np} potential must have a non-central component which is quantitatively explained by S_{12}-term in OPEP.