Chapter 10: Perturbative approach to many-body theory

Ref: Fetter & Walecka, p. 53–116, p. 120–127

Outline:

1. Given many-body Hamiltonian: $\boxed{\hat{H} = \hat{H}_0 + \hat{H}_1}$

   $$\hat{H}_0 |\phi_n\rangle = E_n |\phi_n\rangle \quad \text{"known"}$$

   $\hat{H}_1$ = interaction Hamiltonian

2. "Pictures": Schrödinger (S), interaction (I), and Heisenberg (H)

3. Time-development operator $\hat{U}^{(I)}(t, t_0)$

4. Fermion propagator (Green's function) $G^{(I)}_{\alpha\beta}(\vec{x}, t; \vec{x}', t')$

5. Gell-Mann & Low theorem for exact groundstate

   $$|\psi_0'\rangle = \frac{|\psi_0\rangle}{\langle\phi_0|\psi_0\rangle} = \lim_{\varepsilon\to 0} \frac{\hat{U}_\varepsilon(0,-\infty)|\phi_0\rangle}{\langle\phi_0|\hat{U}_\varepsilon(0,-\infty)|\phi_0\rangle}$$

6. Perturbative expansion of $G_{\alpha\beta}$:

   $$G = G^{(0)} + G^{(1)} + G^{(2)} \dots$$

   derive Feynman rules for $G_{\alpha\beta}$.

7. Calculate observables: a) $E_{g.s.} = E_0 + \Delta E^{(1)} + \Delta E^{(2)} + \dots$ (time-dep. version of Goldstone theorem)

   b) one-body operators

8. Dyson's equations and self-energy insertion

   static HF $\hat{=}$ perturb. theory in $\underline{\underline{\infty}}$ order?

QM "pictures" (= representations): Schrödinger (S), interaction (I) and Heisenberg (H).

Ref: Fetter & Walecka, p. 53–59

We are all familiar with the Schrödinger picture. The fermion propagator is actually defined in the H-picture, and perturbative calculations (Feynman diagrams) are carried out in the I-picture. Hence, we need to discuss all of these.

Brief review of Schrödinger picture (S)

In this chapter, we focus on time-indep. Schrödinger Hamiltonians $\hat{H}_S$.

$$\hat{H}_S \, |\phi_S(t)\rangle = i\hbar \frac{\partial}{\partial t} |\phi_S(t)\rangle \qquad (1)$$

Schrödinger picture

Formal solution (prove by differentiation with respect to t):

$$|\phi_S(t)\rangle = \exp\left[-\frac{i}{\hbar}\hat{H}_S \cdot (t-t_0)\right] |\phi_S(t_0)\rangle, \qquad (2)$$

where the exponential operator is defined via its Taylor series expansion.

We define the time-development operator in the S-picture:

$$\boxed{\hat{U}_S(t,t_0) \overset{def}{=} \exp\left[-\frac{i}{\hbar}\hat{H}_S \cdot (t-t_0)\right]} \qquad (3)$$

resulting in

$$\boxed{|\phi_S(t)\rangle = \hat{U}_S(t,t_0)\, |\phi_S(t_0)\rangle} \quad (4) \quad \left\{ \begin{array}{l} \text{S-picture state} \\ \text{vector is time-} \\ \text{dependent!} \end{array} \right\}$$

Interaction picture (I)

Developed by Paul Dirac. Assume that we can split up total Hamiltonian in the form

$$\boxed{\hat{H}_s = \underset{\substack{\text{"free"}\\\text{(exactly solvable)}}}{\hat{H}_0} + \underset{\text{"interaction"}}{\hat{H}_1^{(S)}}} \qquad (5)$$

<u>Define state vector in I-picture</u>

$$\boxed{|\phi_I(t)\rangle = \exp[\tfrac{i}{\hbar}\hat{H}_0 t]\,|\phi_s(t)\rangle} \qquad (6)$$

One finds, using $i\hbar\frac{\partial}{\partial t}|\phi_s(t)\rangle = \hat{H}_s|\phi_s(t)\rangle = (\hat{H}_0 + \hat{H}_1^{(S)})|\phi_s(t)\rangle$

$$\boxed{\begin{gathered} i\hbar\frac{\partial}{\partial t}|\phi_I(t)\rangle = \hat{H}_1^{(I)}(t)\,|\phi_I(t)\rangle \\ \text{with } \underbrace{\hat{H}_1^{(I)}(t)}_{\text{I-picture}} = e^{i\hat{H}_0 t/\hbar}\,\underbrace{\hat{H}_1^{(S)}}_{\text{S-picture}}\,e^{-i\hat{H}_0 t/\hbar}\end{gathered}} \qquad (7)$$

This explains the notion "interaction picture": the time-development of the state vector $|\phi_I(t)\rangle$ is determined solely by the interaction Hamiltonian $\hat{H}_1$. The effect of $\hat{H}_0$ has be transformed away!

From equs (6) and (2):

$$|\phi_I(t)\rangle \overset{(6)}{=} e^{i\hat{H}_0 t/\hbar}\,|\phi_s(t)\rangle \overset{(2)}{=} e^{i\hat{H}_0 t/\hbar}\,e^{-i\hat{H}_s(t-t_0)/\hbar}\,\underbrace{|\phi_s(t_0)\rangle}_{\overset{(6)}{=} e^{-iH_0 t_0/\hbar}|\phi_I(t_0)\rangle}$$

$$\Rightarrow \boxed{\begin{gathered}|\phi_I(t)\rangle = \hat{U}_I(t,t_0)\,|\phi_I(t_0)\rangle \\ \text{with } \hat{U}_I(t,t_0) = e^{i\hat{H}_0 t/\hbar}\,e^{-i\hat{H}_s(t-t_0)/\hbar}\,e^{-i\hat{H}_0 t_0/\hbar}\end{gathered}} \qquad (8)$$

From expression (8) for $\hat{U}_I$ one finds:

$$\boxed{\begin{array}{l} \text{(1)}\quad \hat{U}_I \text{ is unitary: } \hat{U}_I^+ \hat{U}_I = \hat{U}_I \hat{U}_I^+ = 1 \\ \text{(2)}\quad \text{group property: } \hat{U}_I(t_1,t_2)\,\hat{U}_I(t_2,t_3) = \hat{U}_I(t_1,t_3) \\ \text{(3)}\quad \text{time-reversal property: } \hat{U}_I(t_0,t) = \hat{U}^+(t,t_0) \end{array}} \qquad \underline{(9)}$$

The expression $\underline{(8)}$ for $\hat{U}_I$ is not very useful in practical calculations. It is better to derive an alternate expression starting from eq. (7):

$$i\hbar \frac{\partial}{\partial t} |\phi_I(t)\rangle = \hat{H}_1^{(I)}(t)\, |\phi_I(t)\rangle$$

$$\| (8) \qquad\qquad\qquad\qquad (8)$$

$$i\hbar \frac{\partial}{\partial t} \hat{U}_I(t,t_0)\,|\phi_I(t_0)\rangle = \hat{H}_1^{(I)}(t)\, \hat{U}_I(t,t_0)\,|\phi_I(t_0)\rangle$$

$\Rightarrow$ <u>"operator eq." for $\hat{U}_I$:</u> $\boxed{i\hbar \frac{\partial}{\partial t'} \hat{U}_I(t',t_0) = \hat{H}_1^{(I)}(t')\, \hat{U}_I(t',t_0)}$

where we have replaced $t \to t'$! $\qquad\qquad \underline{(10)}$

We now integrate over t' in the interval $[t_0, t]$. This yields

$$\int_{t_0}^{t} \frac{\partial \hat{U}(t',t_0)}{\partial t'}\, dt' = \frac{1}{i\hbar} \int_{t_0}^{t} dt'\, \hat{H}_1(t')\, \hat{U}(t',t_0)$$

where we have dropped the index (I), for simplicity.

$$\hat{U}(t',t_0)\Big|_{t_0}^{t} = \hat{U}(t,t_0) - \underbrace{\hat{U}(t_0,t_0)}_{=1} = \frac{1}{i\hbar}\int_{t_0}^{t} dt' \ldots$$

$$\Rightarrow \quad \boxed{\hat{U}(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, \hat{H}_1(t')\, \hat{U}(t',t_0)} \qquad \underline{(11)}$$

Solve this eqn. by iteration; on RHS, replace $U(t',t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t'} dt'' H_1(t'')[1-\ldots]$

$$\Rightarrow \hat{U}(t,t_0) = 1 + \left(\frac{-i}{\hbar}\right)\int_{t_0}^{t} dt' \hat{H}_1(t')\left[1 + \left(-\frac{i}{\hbar}\right)\int_{t_0}^{t'} dt'' \hat{H}_1(t'') + \ldots\right]$$

$$\Rightarrow \boxed{\hat{U}(t,t_0) = 1 + \left(\frac{-i}{\hbar}\right)\int_{t_0}^{t} dt' \hat{H}_1(t') + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \hat{H}_1(t')\hat{H}_1(t'') + \ldots} \tag{12}$$

We now split up the 3^{rd} term on the RHS into 2 equal parts

$$T_3 = \underbrace{\frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \hat{H}_1(t')\hat{H}_1(t'')}_{T_3^{(1)}} + \underbrace{\frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \hat{H}_1(t')\hat{H}_1(t'')}_{T_3^{(2)}}$$

In $T_3^{(1)}$, we can replace $\int_{t_0}^{t'} dt'' \rightarrow \int_{t_0}^{t} dt'' \theta(t'-t'')$ which contributes to the integral only for times $t''<t'$. Hence

$$T_3 = T_3^{(1)} + T_3^{(2)} = \frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' H_1(t') H_1(t'') \theta(t'-t'') + T_3^{(2)}.$$

In the second term, $T_3^{(2)}$, one reverses the order of integration and then changes dummy variables $t' \leftrightarrow t''$ (details see FW p. 57). One finds

$$T_3 = \frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \left[\hat{H}_1(t')\hat{H}_1(t'')\theta(t'-t'') + \hat{H}_1(t'')\hat{H}_1(t')\theta(t''-t') \right]$$

We can simplify the notation by introducing the "time-ordered product" (T-product). For the special case of int. Hamiltonian one finds

$$T\left(\hat{H}_1(t')\hat{H}_1(t'')\right) = \begin{cases} \hat{H}_1(t')\hat{H}_1(t''), & \text{if } t'>t''. \\ \hat{H}_1(t'')\hat{H}_1(t'), & \text{if } t''>t'. \end{cases}$$

[note: latest time to the left.]

The general def. of T-product will be given later (FW p. 87).

Using T, the third term in $\hat{U}(t,t_0)$ becomes:

$$\left(-\frac{i}{\hbar}\right)^2 T_3 = \left(-\frac{i}{\hbar}\right)^2 \frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt''\, T[\hat{H}_1(t')\hat{H}_1(t'')].$$

Trivially, we can replace $\hat{H}_1(t')$ in the 2nd term of $\hat{U}$ by $T(\hat{H}_1(t'))$ and obtain

$$\left.\begin{aligned}\hat{U}(t,t_0) = 1 &+ \left(-\frac{i}{\hbar}\right)\int_{t_0}^{t} dt'\, T(\hat{H}_1(t')) + \\ &+ \left(-\frac{i}{\hbar}\right)^2 \frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t} dt''\, T[\hat{H}_1(t')\hat{H}_1(t'')] + \dots\end{aligned}\right\}$$

So, the general result is just a Taylor series of an exponential involving the interaction Hamiltonian:

$$\boxed{\begin{aligned}&\hat{U}(t,t_0) = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \frac{1}{n!}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n\, T[\hat{H}_1^{(I)}(t_1)\dots\hat{H}_1^{(I)}(t_n)] \\ &\text{or, equivalently:} \\ &\hat{U}(t,t_0) = T\left\{\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, \hat{H}_1^{(I)}(t')\right]\right\} \qquad \underline{(13)}\end{aligned}}$$

Heisenberg picture (H)

This picture is <u>closest to classical mechanics</u>, because we have <u>time-dep.</u> canonical variable <u>operators</u>:

$$\hat{x}_H(t) \text{ and } \hat{p}_H(t)$$

but the state vector $|\phi_H\rangle$ is <u>time-independent</u>.

<u>Suggested review:</u> Shankar, QM, p. 490-491: harmonic oscillator in H-picture.

The state vector in the H-picture is defined as

$$|\phi_H(t)\rangle \stackrel{\text{def}}{=} e^{i\hat{H}_S t/\hbar} |\phi_S(t)\rangle \qquad \underline{(14a)}$$

$\uparrow$ $\underline{\text{full}}$ Hamiltonian

Take time-derivative:

$$\underline{i\hbar \frac{\partial}{\partial t} |\phi_H(t)\rangle} = i\hbar \frac{\partial}{\partial t} \left[e^{i\hat{H}_S t/\hbar} |\phi_S(t)\rangle \right] =$$

$$= e^{i\hat{H}_S t/\hbar} \underbrace{\left[-\hat{H}_S + i\hbar \frac{\partial}{\partial t} \right] |\phi_S\rangle}_{\stackrel{(1)}{=}} = \underline{0}$$

$$\curvearrowright \quad \boxed{|\phi_H\rangle = \text{time-independent!}} \qquad \underline{(14b)}$$

One finds the following connection between operators $\hat{O}$ in the H- and I-pictures (see FW p. 59)

$$\boxed{\hat{O}_H(t) = \hat{U}_I(0,t)\, \hat{O}_I(t)\, \hat{U}_I(t,0)} \qquad \underline{(15)}$$

Also note that

$$\underline{|\phi_H\rangle} \stackrel{(14b)}{=\!=} |\phi_H(0)\rangle \stackrel{(14a)}{=\!=} \underline{|\phi_S(0)\rangle} \stackrel{(6)}{=} \underline{|\phi_I(0)\rangle} \qquad \underline{(16)}$$

The fermion propagator $G_{\alpha\beta}$

<u>Steps:</u>

1) Define $G_{\alpha\beta}$ in H-picture
2) transform to I-picture, physical interpretation
3) special case: calculate <u>free</u> fermion propagator $G^0_{\alpha\beta}$.

1) Define fermion propagator (H-picture)

In the S-picture, we have defined field operators for $s=\frac{1}{2}$ fermions which destroy/create particles:

$$\left.\begin{aligned} \hat{\psi}_\alpha^{(S)}(\vec{x}) &= \sum_{k=1}^{\infty} \hat{c}_{k\alpha}\,\varphi_{k\alpha}(\vec{x}) \qquad \boxed{\alpha = \uparrow\downarrow} \\ &\qquad \text{O.N. s.p. basis} \\ \hat{\psi}_\alpha^{(S)+}(\vec{x}) &= \sum_{k=1}^{\infty} \hat{c}_{k\alpha}^{+}\,\varphi_{k\alpha}^{+}(\vec{x}) \end{aligned}\right\} \quad (17)$$

According to FW, eq. (6.28), the transf. between operators from S- to H-picture is given by

$$\hat{O}_H(t) = e^{i\hat{H}_S t/\hbar}\,\hat{O}_S\,e^{-i\hat{H}_S t/\hbar} \qquad (18a)$$

apply to field destruction operator

$$\hat{\psi}_\alpha^{(H)}(\vec{x},t) = e^{i\hat{H}_S t/\hbar}\,\hat{\psi}_\alpha^{(S)}(\vec{x})\,e^{-i\hat{H}_S t/\hbar}. \qquad (18b)$$

Lets $|\phi_0^{(H)}\rangle$ be the time-indep. g.s. of the N-particle system

$$\hat{H}\,|\phi_0^{(H)}\rangle = E\,|\phi_0^{(H)}\rangle. \qquad (19)$$

We now define the fermion propagator as follows

$$i\,G_{\alpha\beta}(\vec{x},t;\vec{x}',t') = \frac{\langle\phi_0^{(H)}|\,T\{\hat{\psi}_\alpha^{(H)}(\vec{x},t)\,(\hat{\psi}_\beta^{(H)})^{+}(\vec{x}',t')\}\,|\phi_0^{(H)}\rangle}{\langle\phi_0^{(H)}|\phi_0^{(H)}\rangle} \qquad (20)$$

i.e. $G_{\alpha\beta}$ is a time-ordered g.s. expectation value for fermion destruction and creation operators, defined in the H-picture.

Here, the T-product definition is slightly generalized:

$$\left.\begin{aligned} &T\{\hat{\psi}_\alpha^{(H)}(\vec{x},t)\,\hat{\psi}_\beta^{(H)+}(\vec{x}',t')\} = \\ &= \theta(t-t')\,\hat{\psi}_\alpha^{(H)}\hat{\psi}_\beta^{(H)+} \underset{\text{fermions}}{\overset{\text{bosons}}{\pm}} \theta(t'-t)\,\hat{\psi}_\beta^{(H)+}\hat{\psi}_\alpha^{(H)} \end{aligned}\right\} \quad (21)$$

Further details about T-product, see FW, p.65

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2) $G_{\alpha\beta}$ in I-picture and physical interpretation

Ref: FW p. 79-80

We consider now the propagator for $t>t'$. From (20) and (21):

$$i\,G_{\alpha\beta} = \frac{\langle \phi_0^{(H)} |\, \hat{\psi}_\alpha^{(H)}(\vec{x},t)\,\hat{\psi}_\beta^{(H)+}(\vec{x}',t')\,| \phi_0^{(H)} \rangle}{\langle \phi_0^{(H)} | \phi_0^{(H)} \rangle}$$

Transform field operators from (H) to (I)-picture, using (15):

$$\hat{\psi}_\alpha^{(H)}(\vec{x},t) = \hat{U}_I(0,t)\,\hat{\psi}_\alpha^{(I)}(\vec{x},t)\,\hat{U}_I(t,0)$$

Inserting above:

$$\left.\begin{aligned} i\,G_{\alpha\beta} = &\underbrace{\langle \phi_0^{(H)} |\, \hat{U}_I(0,t)}\,\hat{\psi}_\alpha^{(I)}(\vec{x},t)\,\overbrace{\hat{U}_I(t,0)\,\hat{U}_I(0,t')}^{=\hat{U}_I(t,t')}\,\hat{\psi}_\beta^{(I)+}(\vec{x}',t') \\ &\cdot \underbrace{\hat{U}_I(t',0)\,|\phi_0^{(H)}\rangle}_{=|\phi_0^{(I)}(t')\rangle} \Big/ \underbrace{\langle \phi_0^{(H)} | \phi_0^{(H)} \rangle}_{=\langle \phi_0^{(I)}(0) | \ldots} \end{aligned}\right\}$$

see (16)!

The underlined terms can be rewritten as

$$\langle \phi_0^{(H)} |\, \hat{U}_I(0,t) = \langle \phi_0^{(I)} |\, \hat{U}_I^+(t,0) = \langle \hat{U}_I(t,0)\,\phi_0^{(I)} | = \langle \phi_0^{(I)}(t) |$$

$$\begin{matrix} \| & \| \\ \langle \phi_0^{(I)} | & \hat{U}_I^+(t,0) \end{matrix}$$

resulting in the <u>expression</u> (<u>I-picture</u>):

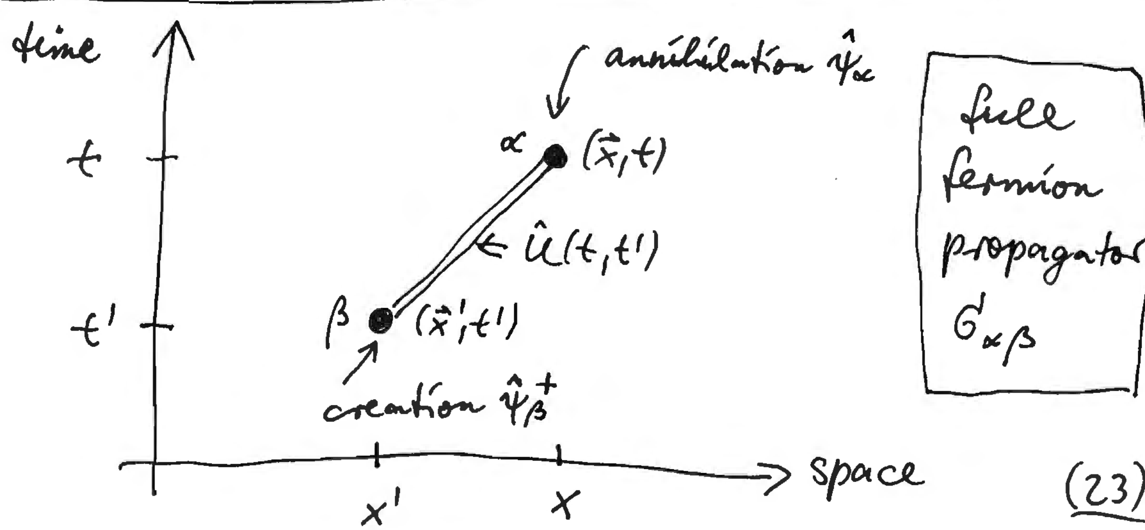
for $t > t'$:

$$i\, G_{\alpha\beta}(\vec{x},t;\vec{x}\,',t') = \frac{\langle \phi_0^{(I)}(t) | \hat{\psi}_\alpha^{(I)}(\vec{x},t)\, \hat{U}_I(t,t')\, \psi_\beta^{(I)+}(\vec{x}\,',t') | \phi_0^{(I)}(t') \rangle}{\langle \phi_0^{(I)}(0) | \phi_0^{(I)}(0) \rangle} \qquad (22)$$

Interpretation of eq. (22); read from right to left:

1. start from <u>N-particle g.s.</u> at time t', $|\phi_0^{(I)}(t')\rangle$.
2. $\hat{\psi}_\beta^{(I)+}(\vec{x}\,',t')$ <u>creates</u> <u>additional</u> <u>fermion</u> (with spin projection β) at position $\vec{x}\,'$ at time t'!
3. the $(N+1)$-particle <u>system "propagates"</u> in time from $t' \to t$, via $\hat{U}_I(t,t')$.
4. at time t and position $\vec{x}$, the additional <u>fermion</u> is <u>destroyed</u> (with spin proj. α), and we return to the N-particle g.s.

In pictorial representation ("Feynman diagram")



(23)

3) Special case: calculate free propagator $G^{(0)}_{\alpha\beta}$

Ref: FW, p. 70-72

In this case we have

interacting fermions	free fermions
$G_{\alpha\beta}$	$G^{(0)}_{\alpha\beta}$
$\hat{H} = \hat{H}_0 + \hat{H}_1$	$\hat{H}_1 = 0$
$\hat{U}(t,t_0)$	$\hat{U} = T[\exp(-i\int \underbrace{\hat{H}_1}_{=0})] = \underline{\underline{1}}$
int. g.s. $\|\phi_0^{(I)}(t)\rangle$	free g.s. $\|\phi_0\rangle$

(24)

Inserting (24) into (22) we obtain

$$\underset{(t>t')}{iG^0_{\alpha\beta}} = \langle\phi_0|\hat{\psi}^{(I)}_\alpha(\vec{x},t)\cdot 1\cdot\hat{\psi}^{(I)+}_\beta(\vec{x}',t')|\phi_0\rangle / \underbrace{\langle\phi_0|\phi_0\rangle}_{=1}$$

This may be generalized to arbitrary times $t \gtrless t'$ via T-product

$$iG^0_{\alpha\beta}(\vec{x},t;\vec{x}',t') = \langle\phi_0|T[\hat{\psi}^{(I)}_\alpha(\vec{x},t)\,\hat{\psi}^{(I)+}_\beta(\vec{x}',t')]|\phi_0\rangle. \quad (25)$$

Use eq. (21) for T-product def.

$$iG^0_{\alpha\beta} = \theta(t-t')\langle\phi_0|\hat{\psi}^{(I)}_\alpha(\vec{x},t)\,\hat{\psi}^{(I)+}_\beta(\vec{x}',t')|\phi_0\rangle$$
$$-\,\theta(t'-t)\langle\phi_0|\hat{\psi}^{(I)+}_\beta(\vec{x}',t')\,\hat{\psi}^{(I)}_\alpha(\vec{x},t)|\phi_0\rangle.$$

note: minus sign for fermions!

We now expand the field operators in a basis. In the S-picture we have (eq. 17)

$$\hat{\psi}^{(S)}_\alpha(\vec{x}) = \sum_{k=1}^{\infty} \hat{c}_{\vec{k}\alpha}\,\varphi_{\vec{k}\alpha}(\vec{x})$$

One can show that the only difference between S-picture and I-picture is that the destruction/creation operators become time-dependent (FW, p. 55):

S-picture	I-picture
$\hat{c}_{\vec{k}\alpha}$	$\hat{c}_{\vec{k}\alpha}(t) = \hat{c}_{\vec{k}\alpha}\, e^{-i\omega_k t}$
$\hat{c}^{+}_{\vec{k}\alpha}$	$\hat{c}^{+}_{\vec{k}\alpha}(t) = \hat{c}^{+}_{\vec{k}\alpha}\, e^{+i\omega_k t}$

$$\rightsquigarrow \hat{\psi}^{(I)}_\alpha(\vec{x},t) = \sum_{k=1}^{\infty} \hat{c}_{\vec{k}\alpha} \underbrace{e^{-i\omega_k t}\varphi_{\vec{k}\alpha}(\vec{x})}_{\stackrel{\text{def}}{=}\ \varphi_{\vec{k}\alpha}(\vec{x},t)}$$

Inserting the plane-wave spinor WF's $\varphi_{\vec{k}\alpha}(\vec{x})$, we obtain

$$\boxed{\varphi_{\vec{k}\alpha}(x,t) = L^{-3/2}\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)}\, \eta_\alpha}$$

$\uparrow$ spinor

and the field operators in the I-picture

$$\hat{\psi}^{(I)}_\alpha(\vec{x},t) = \sum_{k} \hat{c}_{\vec{k}\alpha}\, \varphi_{\vec{k}\alpha}(\vec{x},t).$$

We now carry out the <u>particle-hole transf.</u> discussed in lecture notes ch. 3, p. 2:

$$\boxed{\hat{\psi}^{(I)}_\alpha(\vec{x},t) = \Big[\sum_{\vec{k}>k_F} \hat{a}_{\vec{k}\alpha} + \sum_{\vec{k}<k_F} \hat{b}^{+}_{-\vec{k}\alpha}\Big]\varphi_{\vec{k}\alpha}(\vec{x},t)}$$

particles $\qquad$ holes

Insert this expression into $i\,G^{0}_{\alpha\beta}$ on p. 11:

$$
\begin{cases}
i G^{0}_{\alpha\beta}(\vec{x},t;\vec{x}',t') = \theta(t-t')\langle\phi_0|\Big\{\sum_{\vec{k}>k_F}\hat{a}_{\vec{k}\alpha} + \sum_{\vec{k}<k_F}\hat{b}^{+}_{-\vec{k}\alpha}\Big\}\varphi_{\vec{k}\alpha}(\vec{x},t)\times \\
\times\Big\{\sum_{\vec{\ell}>k_F}\hat{a}^{+}_{\vec{\ell}\beta} + \sum_{\vec{\ell}<k_F}\hat{b}_{-\vec{\ell}\beta}\Big\}\varphi^{+}_{\vec{\ell}\beta}(\vec{x}',t')|\phi_0\rangle - \theta(t'-t)\times \\
\langle\phi_0|\Big\{\sum_{\vec{\ell}>k_F}\hat{a}^{+}_{\vec{\ell}\beta} + \sum_{\vec{\ell}<k_F}\hat{b}_{-\vec{\ell}\beta}\Big\}\varphi^{+}_{\vec{\ell}\beta}(\vec{x}',t')\Big\{\sum_{\vec{k}>k_F}\hat{a}_{\vec{k}\alpha} + \sum_{\vec{k}<k_F}\hat{b}^{+}_{-\vec{k}\alpha}\Big\}\varphi_{\vec{k}\alpha}(\vec{x},t)|\phi_0\rangle
\end{cases}
$$

We multiply out the various terms and use the fact that the <u>free g.s. $|\phi_0\rangle$</u> is the <u>p-h vacuum</u>, i.e.

$$
\left.\begin{aligned}
&\hat{a}_i|\phi_0\rangle = 0 = \hat{b}_i|\phi_0\rangle \\
\Rightarrow\ &\langle\phi_0|\hat{a}^{+}_i = 0 = \langle\phi_0|\hat{b}^{+}_i
\end{aligned}\right\}
$$

$iG^{0}_{\alpha\beta}$ simplifies considerably:

$$
\left.\begin{aligned}
iG^{0}_{\alpha\beta} = \theta(t-t')\langle\phi_0|\sum_{\vec{k},\vec{\ell}>k_F}\hat{a}_{\vec{k}\alpha}\hat{a}^{+}_{\vec{\ell}\beta}\varphi_{\vec{k}\alpha}(\vec{x},t)\varphi^{+}_{\vec{\ell}\beta}(\vec{x}',t')|\phi_0\rangle \\
-\theta(t'-t)\langle\phi_0|\sum_{\vec{k},\vec{\ell}<k_F}\hat{b}_{-\vec{\ell}\beta}\hat{b}^{+}_{-\vec{k}\alpha}\varphi^{+}_{\vec{\ell}\beta}(\vec{x}',t')\varphi_{\vec{k}\alpha}(\vec{x},t)|\phi_0\rangle
\end{aligned}\right\}
$$

We now use the fermion anticommutation relations

$$\hat{a}_i\hat{a}^{+}_k = \delta_{ik} - \hat{a}^{+}_k\hat{a}_i \quad \text{and} \quad \hat{b}_i\hat{b}^{+}_k = \delta_{ik} - \hat{b}^{+}_k\hat{b}_i$$

and use again the properties $\hat{a}_i|\phi_0\rangle = 0 = \hat{b}_i|\phi_0\rangle \Rightarrow$

$$
\left.\begin{aligned}
iG^{0}_{\alpha\beta} = \theta(t-t')\langle\phi_0|\sum_{\vec{k},\vec{\ell}>k_F}\delta_{\alpha\beta}\delta_{\vec{k}\vec{\ell}}\varphi_{\vec{k}\alpha}(\vec{x},t)\varphi^{+}_{\vec{\ell}\beta}(\vec{x}',t')|\phi_0\rangle \\
-\theta(t'-t)\langle\phi_0|\sum_{\vec{k},\vec{\ell}<k_F}\delta_{\alpha\beta}\delta_{\vec{k}\vec{\ell}}\varphi^{+}_{\vec{\ell}\beta}(\vec{x}',t')\varphi_{\vec{k}\alpha}(\vec{x},t)|\phi_0\rangle
\end{aligned}\right\}
$$

Because of $\delta_{\vec{k}\vec{\ell}}$, the double sums collapse, and we obtain

$$
\left.\begin{aligned}
iG^0_{\alpha\beta} = {} & \theta(t-t')\,\delta_{\alpha\beta}\,\langle\Phi_0|\sum_{\vec{k}}\theta(k-k_F)\,\varphi_{\vec{k}\alpha}(\vec{x},t)\,\varphi^+_{\vec{k}\alpha}(\vec{x}',t')|\Phi_0\rangle \\
& -\theta(t'-t)\,\delta_{\alpha\beta}\,\langle\Phi_0|\sum_{\vec{k}}\theta(k_F-k)\,\varphi^+_{\vec{k}\alpha}(\vec{x}',t')\,\varphi_{\vec{k}\alpha}(\vec{x},t)|\Phi_0\rangle
\end{aligned}\right\}
$$

Inserting the s.p. WF's from p. 12, we arrive at

$$
\left.\begin{aligned}
iG^0_{\alpha\beta} = {} & \frac{\delta_{\alpha\beta}}{L^3}\sum_{\vec{k}}\exp\left[i\left\{\vec{k}\cdot(\vec{x}-\vec{x}')-\omega_k(t-t')\right\}\right]\times \\
& \times\left[\theta(t-t')\,\theta(k-k_F)-\theta(t'-t)\,\theta(k_F-k)\right]
\end{aligned}\right\}
$$

As we have done many times before, we replace $\sum_{\vec{k}} \to \int d^3k$ using FW, eq. (3.26):

$$
\frac{1}{L^3}\sum_{\vec{k}} \to \frac{1}{(2\pi)^3}\int d^3k
$$

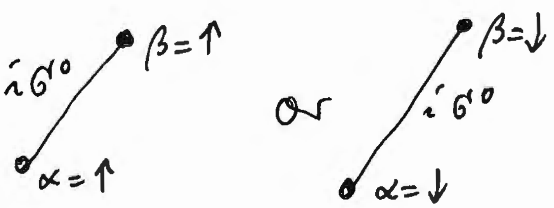
resulting in the final expression for the free fermion propagator

$$
\boxed{
\begin{aligned}
iG^0_{\alpha\beta}(\vec{x}-\vec{x}',t-t') = {} & \delta_{\alpha\beta}\left(\frac{1}{2\pi}\right)^3\int d^3k\, e^{i[\vec{k}\cdot(\vec{x}-\vec{x}')-\omega_k(t-t')]}\cdot \\
& \cdot\left[\theta(t-t')\,\theta(k-k_F)-\theta(t'-t)\,\theta(k_F-k)\right]
\end{aligned}
} \quad (26)
$$

FW, (7.42)

Note the $\delta_{\alpha\beta}$-factor for the spin projections $(\uparrow\downarrow)$. This means that G^0 is diagonal in spin-space, i.e.

$$
iG^0 = iG^0_{\alpha\beta} = i\begin{pmatrix} G^0_{\uparrow\uparrow} & G^0_{\uparrow\downarrow} \\ G^0_{\downarrow\uparrow} & G^0_{\downarrow\downarrow}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\left(\frac{1}{2\pi}\right)^3\int d^3k \ldots
$$



This makes sense, because a "spin-flip" component such as $G^0_{\uparrow\downarrow}$ or $G^0_{\downarrow\uparrow}$ would require a spin-dep. interaction (but here the fermion is free).

Fermion propagator $\hat{=}$ Green's function

One can prove (FW p. 117) that iG^0 for an N-fermion system obeys the wave eqn

$$\boxed{\left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\right] G^0_{\alpha\beta}(\vec{x}-\vec{x}', t-t') = \hbar\, \delta_{\alpha\beta}\, \delta^3(\vec{x}-\vec{x}')\, \delta(t-t').}$$

i.e. the RHS of this eqn is a point source in space and time, which is of the same form as Green's functions in E&M, see e.g. Jackson, 3rd ed. p. 612:

$$\boxed{\begin{gathered}\Box_x D(x-x') = \delta^{(4)}(x-x') \\ \text{with } \Box_x = \partial_\alpha \partial^\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\end{gathered}}$$

Construction of the exact g.s. in perturbation theory: the Gell-Mann and Low theorem (1951)

Ref: FW, p. 59–64

We want to solve the static many-body problem for the ground state; in the S-picture we have

$$\hat{H}_S |\Psi_S\rangle = E |\Psi_S\rangle$$

exact g.s. energy / state vector

We presume that we can split up $\hat{H}_S$ in the form

$$\hat{H}_S = \underbrace{\hat{H}_0}_{\text{non-int.}} + \hat{H}_1$$

and that we can construct the non-interacting g.s.

$$\hat{H}_0 |\phi_0\rangle = E_0 |\phi_0\rangle .$$

The basic idea of Gell-Mann & Low was to transform this static problem into a time-dep. problem by "adiabatically switching on" the interaction. The problem is then solved using $\hat{U}(t,t_0)$ in the I-picture:

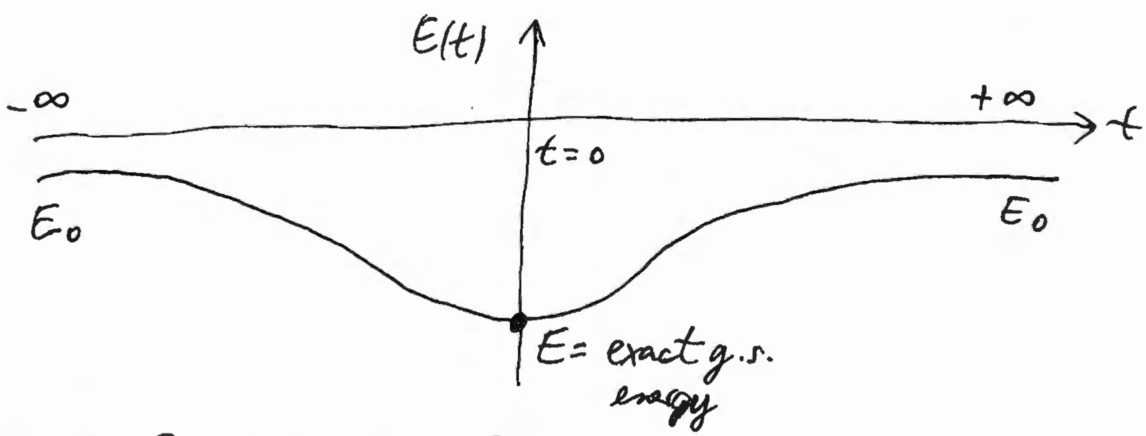
$$\hat{H}_S(\text{static}) = \hat{H}_0 + \hat{H}_1 \longrightarrow \boxed{\hat{H}_S(t) = \hat{H}_0 + e^{-\varepsilon|t|}\hat{H}_1 \quad \left(\begin{array}{c}\text{time-}\\ \text{dep.}\end{array}\right)}$$

where ε = small, positive constant. We observe

$$\hat{H}_S(t \to \pm\infty) = \hat{H}_0 \quad (\text{non-int. system})$$

$$H_S(t=0) = \hat{H}_0 + \hat{H}_1 = \text{full Hamiltonian (static)}$$

corresponding energies as function of time:



In the S-picture we have

$$\left(i\hbar \frac{\partial}{\partial t} - \hat{H}_S(t)\right) |\psi_S(t)\rangle = 0$$

In the limit $t_0 \to -\infty$, one finds

$$\left(i\hbar \frac{\partial}{\partial t_0} - \hat{H}_0\right) |\psi_S(t_0 \to -\infty)\rangle = 0$$

From $H_0 |\phi_0\rangle = E_0 |\phi_0\rangle$ we find: $|\psi_S(t_0 \to -\infty)\rangle = |\phi_0\rangle e^{-i E_0 t_0/\hbar}$.

Transform S-picture → I-picture:

$$|\psi_I(t_0\to-\infty)\rangle = e^{i\hat{H}_0 t_0/\hbar}|\psi_S(t_0\to-\infty)\rangle =$$

$$= \underbrace{e^{i\hat{H}_0 t_0/\hbar}|\phi_0\rangle}\, e^{-iE_0 t_0/\hbar}$$

$$= e^{iE_0 \cancel{t_0/\hbar}}|\phi_0\rangle e^{-iE_0\cancel{t_0/\hbar}} = |\phi_0\rangle\,, \quad (*)$$

i.e. this state vector is time-independent.

On page 6', we derived the time-development operator for static $\hat{H}_1^{(S)}$:

$$\hat{U}^{(I)}(t,t_0) = T\left\{\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\,\hat{H}_1^{(I)}(t')\right]\right\}$$

For time-dep. $\hat{H}_1^{(S)} \to \hat{H}_1^{(S)} e^{-i\varepsilon|t|}$ one finds

$$\boxed{\hat{U}_\varepsilon^{(I)}(t,t_0) = T\left\{\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\,e^{-\varepsilon|t'|}\hat{H}_1^{(I)}(t')\right]\right\}} \quad (**)$$

The g.s. in the H-picture, which appears in the definition of G', is found to be

$$|\psi_H\rangle \overset{p.7}{=\!=} |\psi_I(0)\rangle = \hat{U}_\varepsilon^{(I)}(0,t_0)|\psi_I(t_0)\rangle$$

Taking the limit $t_0\to-\infty$ and using eq.(*) above we find

$$\boxed{|\psi_H\rangle = \hat{U}_\varepsilon^{(I)}(0,-\infty)|\psi_I(-\infty)\rangle \overset{(*)}{=} \hat{U}_\varepsilon^{(I)}(0,-\infty)|\phi_0\rangle} \quad (***)$$

Suppose now that we carry out this procedure in perturb. theory using $\hat{U}_\varepsilon^{(I)}(t,t_0)$. The result depends on the small parameter ε. The true physical limit is obtained for $\varepsilon\to0$. Gell-Mann & Low found the surprising result that the above quantity

$$\boxed{|\psi_H\rangle = |\psi_I(0)\rangle \overset{def}{=} |\psi_0\rangle = \hat{U}_\varepsilon^{(I)}(0,-\infty)|\phi_0\rangle}$$

does not exist in the limit $\varepsilon \to 0_+$, in fact it diverges like $\frac{1}{\varepsilon}$.

However, the closely related quantity

$$\boxed{|\psi_0'\rangle \overset{def}{=} \frac{|\psi_0\rangle}{\langle\phi_0|\psi_0\rangle} = \lim_{\varepsilon\to 0} \frac{\hat{U}_\varepsilon^{(I)}(0,-\infty)|\phi_0\rangle}{\langle\phi_0|\hat{U}_\varepsilon^{(I)}(0,-\infty)|\phi_0\rangle}} \qquad (1)$$

exists (provided that the system has a perturb. series to all orders n), and the Gell-Mann & Low theorem states that in this case $|\psi_0'\rangle$ is the exact g.s:

Gell-Mann & Low theorem:

The state vector (1) represents the exact g.s. of the system:

$$\underbrace{\hat{H}|\psi_0'\rangle}_{\substack{\text{exact g.s.}\\ \text{state vector}}} = \underbrace{E}_{\substack{\text{exact g.s.}\\ \text{energy}}}|\psi_0'\rangle \qquad (2)$$

The purpose of the denominator in eq.(1) is to cancel out the $\frac{1}{\varepsilon}$ divergence as $\varepsilon \to 0$. The proof of the theorem is very technical, see FW p. 61-64.

Derive expression for g.s. energy shift $E - E_0$

Insert eq.(1) into eq.(2):

$$\frac{\hat{H}|\psi_0\rangle}{\langle\phi_0|\psi_0\rangle} = \frac{E|\psi_0\rangle}{\langle\phi_0|\psi_0\rangle} \quad \Big| \times \langle\phi_0|$$

$$\frac{\langle\phi_0|\hat{H}_0+\hat{H}_1|\psi_0\rangle}{\langle\phi_0|\psi_0\rangle} = \frac{E\cancel{\langle\phi_0|\psi_0\rangle}}{\cancel{\langle\phi_0|\psi_0\rangle}} = E$$

Because $H_0|\phi_0\rangle = E_0|\phi_0\rangle$ on the left, we obtain

$$\frac{E_0\cancel{\langle\phi_0|\psi_0\rangle}}{\cancel{\langle\phi_0|\psi_0\rangle}} + \frac{\langle\phi_0|\hat{H}_1|\psi_0\rangle}{\langle\phi_0|\psi_0\rangle} = E$$

$$\Rightarrow \quad E - E_0 = \frac{\langle\phi_0|\hat{H}_1|\psi_0\rangle}{\langle\phi_0|\psi_0\rangle}$$

This leads to the

Gell-Mann & Low theorem for g.s. energy shift: see FW eq. (6.45) and eq. (9.50)

$$E = E_0 + \Delta E$$

$$\Delta E = E - E_0 = \frac{\langle\phi_0|\hat{H}_1|\psi_0\rangle}{\langle\phi_0|\psi_0\rangle} = \lim_{\varepsilon\to 0}\frac{\langle\phi_0|\hat{H}_1\,\hat{U}_\varepsilon(0,-\infty)|\phi_0\rangle}{\langle\phi_0|\hat{U}_\varepsilon(0,-\infty)|\phi_0\rangle}$$

where $\hat{U}_\varepsilon$ is defined in lecture notes p. ~~[illegible]~~ 17.

Perturbation expansion of exact propagator (using Gell-Mann & Low theorem)

Lit: FW p. 83–85

We start from the def. of the exact propagator in the Heisenberg picture (lecture notes p. ~~[illegible]~~ 8 and FW p. 64)

$$i\,G_{\alpha\beta}(\vec{x}t,\vec{x}'t') = \frac{\langle\psi_0^{(H)}|T[\hat{\psi}_{H\alpha}(\vec{x},t)\hat{\psi}^{+}_{H\beta}(\vec{x}'t')]|\psi_0^{(H)}\rangle}{\langle\psi_0^{(H)}|\psi_0^{(H)}\rangle}$$

where the Heisenberg g.s. may be expressed as (lecture notes p. ~~[illegible]~~ 18):

$$\underline{|\psi_0^{(H)}\rangle = |\psi_I(0)\rangle \equiv |\psi_0\rangle = U_\varepsilon^{(I)}(0,-\infty)|\phi_0\rangle.}$$

The trick is now to use the Gell-Mann & Low theorem for the g.s.

$$|\psi_0'\rangle = \frac{|\psi_0\rangle}{\langle\phi_0|\psi_0\rangle} = \lim_{\varepsilon\to 0} \frac{\hat{U}_\varepsilon(0,\pm\infty)|\phi_0\rangle}{\langle\phi_0|\hat{U}_\varepsilon(0,\pm\infty)\phi_0\rangle} \qquad (1)$$

To do this, we divide the numerator and denominator of the propagator by $|\langle\phi_0|\psi_0\rangle|^2$ and obtain

$$i\, G_{\alpha\beta}(\vec{x},t;\vec{x}'t') = \frac{N}{D} \text{ with}$$

$$N = \langle\psi_0'|T[\hat{\psi}_{H\alpha}(\vec{x},t)\,\hat{\psi}^+_{H\beta}(\vec{x}'t')]|\psi_0'\rangle = \frac{\langle\psi_0|T[\hat{\psi}_{H\alpha}\hat{\psi}^+_{H\beta}]|\psi_0\rangle}{|\langle\phi_0|\psi_0\rangle|^2}$$

well-defined, converges!

and

$$D = \langle\psi_0'|\psi_0'\rangle = \frac{\langle\psi_0|\psi_0\rangle}{|\langle\phi_0|\psi_0\rangle|^2} \qquad (2)$$

Consider now the denominator D; we will later take the limit $\varepsilon\to 0$:

$$D = \frac{\langle\phi_0|\overbrace{\hat{U}^+_\varepsilon(0,+\infty)}^{=\hat{U}_\varepsilon(+\infty,0)}\hat{U}_\varepsilon(0,-\infty)|\phi_0\rangle}{|\langle\phi_0|\hat{U}_\varepsilon(0,-\infty)|\phi_0\rangle|^2} = \frac{\langle\phi_0|\hat{U}_\varepsilon(+\infty,-\infty)|\phi_0\rangle}{|\ldots|^2}$$

Defining the scattering matrix operator $\hat{S}$ as

$$\boxed{\hat{S} \stackrel{\text{def.}}{=} \hat{U}_\varepsilon(+\infty,-\infty)}$$

we obtain

$$\boxed{D = \frac{\langle\phi_0|\hat{S}|\phi_0\rangle}{|\langle\phi_0|\hat{U}_\varepsilon(0,-\infty)|\phi_0\rangle|^2}}$$

Using $|\Psi_0\rangle = \hat{U}_\varepsilon(0,\pm\infty)|\Phi_0\rangle$, the numerator can be written in the form

$$N = \frac{\langle\Psi_0| T[\hat{\Psi}_{H\alpha}\hat{\Psi}^+_{H\beta}]|\Psi_0\rangle}{|\langle\Phi_0|\Psi_0\rangle|^2} \Longrightarrow$$

$$N = \frac{\langle\Phi_0|\hat{U}_\varepsilon(+\infty,0)\,T[\hat{\Psi}_{H\alpha}\hat{\Psi}^+_{H\beta}]\,\hat{U}_\varepsilon(0,-\infty)|\Phi_0\rangle}{|\langle\Phi_0|\hat{U}_\varepsilon(0,-\infty)|\Phi_0\rangle|^2}$$

which leads to

$$\boxed{i\,G_{\alpha\beta} = \frac{N}{D} = \frac{\langle\Phi_0|\hat{U}_\varepsilon(+\infty,0)\,T[\hat{\Psi}_{H\alpha}(\vec{x},t)\hat{\Psi}^+_{H\beta}(\vec{x}',t')]\,\hat{U}_\varepsilon(0,-\infty)|\Phi_0\rangle}{\langle\Phi_0|\hat{S}|\Phi_0\rangle}}$$

where the quantity $|\langle\Phi_0|\hat{U}_\varepsilon(0,-\infty)|\Phi_0\rangle|^2$ cancels out in both N and D.

Two steps are necessary:

1) transform Heisenberg field operators → I-picture

$$\hat{\Psi}_{H\alpha}(\vec{x},t) = \hat{U}^{(I)}_\varepsilon(0,t)\,\hat{\Psi}_{I\alpha}(\vec{x},t)\,\hat{U}^{(I)}_\varepsilon(t,0)$$

2) insert the structure of the time-development operator (p.17):

$$\hat{U}_\varepsilon^{(I)}(t,t_0) = T\left\{\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt'\, e^{-\varepsilon|t'|}\,\hat{H}_1^{(I)}(t')\right]\right\}$$

After some lengthy math manipulations, one arrives at (FW p.85)

$$\boxed{i\,G_{\alpha\beta}(\vec{x},t;\vec{x}',t') = \frac{\langle\Phi_0|T\{\exp[-\frac{i}{\hbar}\int_{-\infty}^{+\infty} d\tau\,\hat{H}_1^{(I)}(\tau)]\,\hat{\Psi}_{I\alpha}(\vec{x},t)\,\hat{\Psi}^+_{I\beta}(\vec{x}',t')\}|\Phi_0\rangle}{\langle\Phi_0|\hat{S}|\Phi_0\rangle}}$$

The interaction Hamiltonian $\hat{H}_1^{(I)}(\tau)$ in I-picture is obtained from S-picture Hamiltonian $\hat{H}_1^{(S)}$ via the transformation

$$\hat{H}_1^{(I)}(\tau) = \underbrace{e^{+i\hat{H}_0\tau/\hbar}}\ \hat{H}_1^{(S)}\ \underbrace{e^{-i\hat{H}_0\tau/\hbar}}$$

$\downarrow$ assume arbitrary 2-body int.

$$\Rightarrow \left\{ \begin{aligned} \hat{H}_1^{(I)}(\tau) &= \frac{1}{2}\sum_{\lambda\lambda'}\sum_{\mu\mu'} \int d^3x \int d^3x' \, \underbrace{e^{i\hat{H}_0\tau/\hbar}}\, \hat{\psi}_\lambda^{(S)+}(\vec{x})\, \hat{\psi}_\mu^{(S)+}(\vec{x}') \times \\ &\times V(\vec{x},\vec{x}')_{\lambda\lambda'\mu\mu'}\, \hat{\psi}_{\mu'}^{(S)}(\vec{x}')\, \hat{\psi}_{\lambda'}^{(S)}(\vec{x})\, \underbrace{e^{-i\hat{H}_0\tau/\hbar}}. \end{aligned} \right. \quad \underline{(1)}$$

Inserting

$$1 = e^{-i\hat{H}_0\tau/\hbar}\ e^{+i\hat{H}_0\tau/\hbar}$$

between the field operators, we obtain the <u>field operators in the I-picture</u> $\hat{\psi}_\lambda^{(I)+}, \ldots$; one finds

$$\left. \begin{aligned} \hat{H}_1^{(I)}(\tau) &= \frac{1}{2}\sum_{\lambda\lambda'\mu\mu'} \int d^3x \int d^3x' \, \hat{\psi}_\lambda^{(I)+}(\vec{x},\underline{\underline{\tau}})\, \hat{\psi}_\mu^{(I)+}(\vec{x}',\underline{\underline{\tau}}) \\ &\times V(\vec{x},\vec{x}')_{\lambda\lambda'\mu\mu'}\, \hat{\psi}_{\mu'}^{(I)}(\vec{x}',\tau)\, \hat{\psi}_{\lambda'}^{(I)}(\vec{x},\tau). \end{aligned} \right\} \quad \underline{(2)}$$

We now compute the time-integral of $\hat{H}_1^{(I)}(\tau)$ which appears in the expression for the propagator $iG_{\alpha\beta}$ on p. 21:

$$\left. \begin{aligned} \underbrace{\int_{-\infty}^{+\infty}} d\tau\, \hat{H}_1^{(I)}(\tau) &= \frac{1}{2}\sum_{\lambda\lambda'\mu\mu'} \int d^3x \underbrace{\int_{-\infty}^{+\infty} d\tau} \int d^3x'\, V(\vec{x},\vec{x}')_{\lambda\lambda'\mu\mu'} \\ &\times \left[\hat{\psi}_\lambda^{(I)+}(\vec{x},\tau)\, \hat{\psi}_\mu^{(I)+}(\vec{x}',\tau)\, \hat{\psi}_{\mu'}^{(I)}(\vec{x}',\tau)\, \hat{\psi}_{\lambda'}^{(I)}(\vec{x},\tau) \right], \end{aligned} \right\} \quad \underline{(3)}$$

where we have assumed that the 2-body potential <u>$V(\vec{x},\vec{x}')$</u> does <u>not contain any derivative operators</u> and hence commutes with the field operators $\hat{\psi}, \hat{\psi}^+$.

This can be <u>written</u> in a <u>more symmetric form</u> by introducing an <u>additional δ-function in time, $\int d\tau'\,\delta(\tau-\tau')$</u>:

$$\left\{ \int_{-\infty}^{+\infty} \hat{H}_1^{(I)}(\tau)\, d\tau = \frac{1}{2} \sum_{\lambda\lambda'\mu\mu'} \int d^3x \int d\tau \int d^3x' \underline{\int d\tau' \delta(\tau-\tau')} \times \right.$$

$$\times V(\vec{x},\vec{x}')_{\lambda\lambda'\mu\mu'} \left[\hat{\psi}_\lambda^{(I)+}(\vec{x},\tau)\, \hat{\psi}_\mu^{(I)+}(\vec{x}',\underline{\tau'})\, \hat{\psi}_{\mu'}^{(I)}(\vec{x}',\underline{\tau'})\, \hat{\psi}_{\lambda'}^{(I)}(\vec{x},\tau) \right]$$

Using the 4-dim. notation

$$x = (\vec{x},\tau) \qquad x' = (\vec{x}',\tau')$$

we can combine the space + time integrals into $\int d^4x \int d^4x'$:

$$\boxed{\int_{-\infty}^{+\infty} \hat{H}_1^{(I)}(\tau)\, d\tau = \frac{1}{2} \sum_{\lambda\lambda'\mu\mu'} \int d^4x \int d^4x' \underbrace{\delta(\tau-\tau') V(\vec{x},\vec{x}')_{\lambda\lambda'\mu\mu'}}_{\stackrel{\text{def}}{=} U(x,x')_{\lambda\lambda'\mu\mu'}} \times \left[\hat{\psi}_\lambda^{(I)+}(x)\, \hat{\psi}_\mu^{(I)+}(x')\, \hat{\psi}_{\mu'}^{(I)}(x')\, \hat{\psi}_{\lambda'}^{(I)}(x) \right]} \qquad \underline{(4a)}$$

where we have defined the "instantaneous 4-potential"
(no time-retardation in non-rel. physics!)

$$\boxed{U(x,x')_{\lambda\lambda'\mu\mu'} \stackrel{\text{def}}{=} \delta(\tau-\tau') V(\vec{x},\vec{x}')_{\lambda\lambda'\mu\mu'}} \qquad \underline{(4b)}$$

$$\boxed{\text{Calculate propagator } G_{\alpha\beta} \text{ in } 1^{st}\text{-order perturbation theory}}$$

Recall from p. 21 (leave out I-picture super/subscripts):

$$i\, \underset{\substack{\nearrow \\ 4\text{-D}}}{G_{\alpha\beta}(x,y)} = \frac{\langle \phi_0 | T\{ \exp[-\frac{i}{\hbar}\int_{-\infty}^{+\infty} d\tau\, \hat{H}_1(\tau)]\, \hat{\psi}_\alpha(x)\hat{\psi}_\beta^+(y)\} | \phi_0 \rangle}{\langle \phi_0 | \underbrace{T\{ \exp[-\frac{i}{\hbar}\int_{-\infty}^{+\infty} d\tau\, \hat{H}_1(\tau)\}}_{=\hat{S}} | \phi_0 \rangle}$$

Expanding the exponential operator in 1^{st} order: $\boxed{e^x = 1 + x}$

we find

$$i\,G_{\alpha\beta}(x,y) = \frac{\langle\Phi_0|T\{\hat\psi_\alpha(x)\hat\psi^+_\beta(y)\}|\Phi_0\rangle - \frac{i}{\hbar}\langle\Phi_0|T\{\int_{-\infty}^{+\infty}d\tau\,\hat H_1(\tau)\,\hat\psi_\alpha(x)\hat\psi^+_\beta(y)\}|\Phi_0\rangle}{\langle\Phi_0|\Phi_0\rangle - \frac{i}{\hbar}\langle\Phi_0|T\{\int_{-\infty}^{+\infty}d\tau\,\hat H_1(\tau)\}|\Phi_0\rangle}$$

which can be written in the form

$$\boxed{i\,G_{\alpha\beta}(x,y) = \frac{i\,G^0_{\alpha\beta} + i\,\overline{G}^{(1)}}{1 + i\,D^{(1)}}} \qquad \text{(up to 1}^{\text{st}}\text{ order)}$$

with the abbreviations

$$i\,\overline{G}^{(1)} = -\frac{i}{\hbar}\langle\Phi_0|T\{\int_{-\infty}^{+\infty}d\tau\,\hat H_1(\tau)\,\hat\psi_\alpha(x)\hat\psi^+_\beta(y)\}|\Phi_0\rangle$$

$$i\,D^{(1)} = -\frac{i}{\hbar}\langle\Phi_0|T\{\int_{-\infty}^{+\infty}d\tau\,\hat H_1(\tau)\}|\Phi_0\rangle$$

Inserting $\int_{-\infty}^{+\infty}d\tau\,\hat H_1(\tau)$ from p. 23, we obtain

$$\boxed{\begin{aligned} i\,\overline{G}^{(1)}(x,y) &= \left(-\frac{i}{\hbar}\right)\frac{1}{2}\sum_{\lambda\lambda'\mu\mu'}\int d^4x_1\int d^4x_1'\,U(x_1,x_1')_{\lambda\lambda'\mu\mu'}\,\times \\ &\times\langle\Phi_0|T\{\hat\psi^+_\lambda(x_1)\hat\psi^+_\mu(x_1')\hat\psi_{\mu'}(x_1')\hat\psi_{\lambda'}(x_1)\hat\psi_\alpha(x)\hat\psi^+_\beta(y)\}|\Phi_0\rangle\end{aligned}}$$

and

$$\boxed{\begin{aligned} i\,D^{(1)} &= \left(-\frac{i}{\hbar}\right)\frac{1}{2}\sum_{\lambda\lambda'\mu\mu'}\int d^4x_1\int d^4x_1'\,U(x_1,x_1')_{\lambda\lambda'\mu\mu'}\,\times \\ &\times\langle\Phi_0|T\{\hat\psi^+_\lambda(x_1)\hat\psi^+_\mu(x_1')\hat\psi_{\mu'}(x_1')\hat\psi_{\lambda'}(x_1)\}|\Phi_0\rangle\end{aligned}}$$

Observe: the calculation of fermion propagator in 1st-order perturb. theory involves the calculation of g.s. expectation values of time-ordered products of field operators $\rightsquigarrow$

use Wick's theorem!

$\boxed{\text{Exact propagator } i\,G_{\alpha\beta}(x,y) \text{ in first-order perturbation theory; Wick's theorem} \to \text{Feynman diagrams in coordinate space}}$

Lit: FW, p. 87–99

From lecture notes p. 24 we find that the numerator of the exact propagator $i\,G_{\alpha\beta}$, in 1st-order perturbation theory, contains the term

$$i\,\tilde{G}^{(1)} \propto \langle \Phi_0 | T\{ \hat{\psi}^+_{\lambda}(x_1)\hat{\psi}^+_{\mu}(x_1')\hat{\psi}_{\mu'}(x_1')\hat{\psi}_{\lambda'}(x_1)\hat{\psi}_{\alpha}(x)\hat{\psi}^+_{\beta}(y)\} |\Phi_0\rangle$$

In applying Wick's theorem, we need to calculate all possible contractions of field operators.

We remind ourselves of the definition of contraction

$$\underbrace{UV} = T(UV) - N(UV) \qquad \text{FW}(8.21)$$

Take g.s. expectation value

$$\langle \Phi_0 | \underbrace{UV} | \Phi_0\rangle = \langle \Phi_0 | T(UV) | \Phi_0\rangle - \underbrace{\langle \Phi_0 | N(UV) | \Phi_0\rangle}_{=0}$$

contraction = c-number!

$$\underbrace{UV}\,\underbrace{\langle \Phi_0|\Phi_0\rangle}_{=1} = \langle \Phi_0 | T(UV) | \Phi_0\rangle \Rightarrow \boxed{\underbrace{UV} = \langle \Phi_0 | T(UV) | \Phi_0\rangle} \quad \text{FW}(8.28)$$

Use last eq. to calculate contraction between $\hat{\psi}_\alpha$ and $\hat{\psi}^+_\beta$:

$$\boxed{\underbrace{\hat{\psi}_\alpha(x)\hat{\psi}^+_\beta(y)} = \langle \Phi_0 | T\{\hat{\psi}_\alpha(x)\hat{\psi}^+_\beta(y)\} | \Phi_0\rangle = i\,G^0_{\alpha\beta}(x,y)} \quad (1)$$

p. 24

Contraction = free propagator!

From the property, FW (8.31)

$$\underbrace{UV}_{} = \overset{\text{bosons}}{\underset{\text{fermions}}{\pm}} \underbrace{VU}_{}$$

we obtain for the fermion field operator contraction between $\hat{\psi}^+$ and $\hat{\psi}$:

$$\boxed{\hat{\psi}^+ \hat{\psi} = - \hat{\psi}\,\hat{\psi}^+ = -(i\,G^0)} \qquad (2)$$

Finally, one can prove that all other possible contractions vanish (FW p. 88), i.e.

$$\boxed{\hat{\psi}_\alpha^+(x)\,\hat{\psi}_\beta^+(y) = 0 = \hat{\psi}_\alpha(x)\,\hat{\psi}_\beta(y)} \qquad (3)$$

Consider now

$i\,\bar{G}^{(1)} \propto \langle \phi_0 | T\{\cdots\} | \phi_0 \rangle =$ use Wick theorem, full contractions only, use (3)

$$= N\{\hat{\psi}_\lambda^+(x_1)\,\hat{\psi}_\mu^+(x_1')\,\hat{\psi}_{\mu'}(x_1')\,\hat{\psi}_{\lambda'}(x_1)\,\hat{\psi}_\alpha(x)\,\hat{\psi}_\beta^+(y)\} + \quad (A)$$

$$N\{\hat{\psi}_\lambda^+(x_1)\,\hat{\psi}_\mu^+(x_1')\,\hat{\psi}_{\mu'}(x_1')\,\hat{\psi}_{\lambda'}(x_1)\,\hat{\psi}_\alpha(x)\,\hat{\psi}_\beta^+(y)\} + \quad (B)$$

$$N\{\hat{\psi}_\lambda^+(x_1)\,\hat{\psi}_\mu^+(x_1')\,\hat{\psi}_{\mu'}(x_1')\,\hat{\psi}_{\lambda'}(x_1)\,\hat{\psi}_\alpha(x)\,\hat{\psi}_\beta^+(y)\} + \quad (C)$$

+ 3 additional terms, see FW, p. 92, eq. (9.1)

We use now the sign-rules for handling contractions inside an N-product, ch. 4, p. 12. In the first term Ⓐ, <u>no rearrangement of field operators is necessary</u>. Remember that <u>contractions are c-numbers</u>, hence their <u>sequence does not matter</u>:

$$
(A) = \hat\psi_\alpha(x)\,\hat\psi_\beta^+(y)\;\hat\psi_\mu^+(x_1')\,\hat\psi_{\mu'}(x_1')\;\hat\psi_\lambda^+(x_1)\,\hat\psi_{\lambda'}(x_1)
$$

Remembering that $i\,G^0_{\delta\gamma} = \hat\psi_\delta\,\hat\psi^+_\gamma$, see eq. (40) and

$\hat\psi^+\hat\psi \overset{(2)}{=} -\hat\psi\hat\psi^+ = -(i\,G^0)$ for fermion operators, we find:

$$
(A) = i\,G^0_{\alpha\beta}(x,y)\,(-i)\,G^0_{\mu'\mu}(x_1',x_1')\,(-i)\,G^0_{\lambda'\lambda}(x_1,x_1). \qquad (49)
$$

In the second term, Ⓑ, we can take out $G^0_{\alpha\beta}(x,y)$ immediately, but we have to rearrange $\hat\psi^+_\mu(x_1')\hat\psi_{\mu'}(x_1')$ inside the N-product which yields a minus-sign:

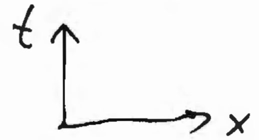
$$
(B) = i\,G^0_{\alpha\beta}(x,y)\,(-1)\,\hat\psi_\lambda^+(x_1)\,\hat\psi_{\mu'}(x_1')\;\hat\psi_\mu^+(x_1')\,\hat\psi_{\lambda'}(x_1) \Longrightarrow
$$

$$
(B) = i\,G^0_{\alpha\beta}(x,y)\,(-1)\,(-i)\,G^0_{\mu'\lambda}(x_1',x_1)\,(-i)\,G^0_{\lambda'\mu}(x_1,x_1'). \qquad (50)
$$

The term Ⓒ can be evaluated in a similar manner. Let us insert the results for Ⓐ, Ⓑ etc. into eq. on p.24 for the <u>numerator of $G_{\alpha\beta}$</u>:

$$
\left.
\begin{aligned}
i\,\bar G^{(1)} = {} & \frac{1}{2}\left(-\frac{i}{\hbar}\right)\sum_{\lambda\lambda'\mu\mu'}\int d^4x_1 \int d^4x_1'\; U(x_1,x_1')_{\lambda\lambda'\mu\mu'}\,\times \\
& \times \Big\{ i\,G^0_{\alpha\beta}(x,y)\, i\,G^0_{\mu'\mu}(x_1',x_1')\, i\,G^0_{\lambda'\lambda}(x_1,x_1) && (A) \\
& \quad -i\,G^0_{\alpha\beta}(x,y)\, i\,G^0_{\mu'\lambda}(x_1',x_1)\, i\,G^0_{\lambda'\mu}(x_1,x_1') && (B) \\
& \quad +i\,G^0_{\alpha\lambda}(x,x_1)\, i\,G^0_{\lambda'\mu}(x_1,x_1')\, i\,G^0_{\mu'\beta}(x_1',y) && (C) \\
& \quad + 3 \text{ add. terms, see FW eq. (19.1)} \Big\}
\end{aligned}
\right\} \qquad (51)
$$

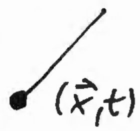
Components of Feynman diagrams



convention: vertical axis $\hat{=}$ time, horizontal axis $\hat{=}$ space

(a)  solid line with <u>upward</u> arrow $\hat{=}$ <u>particle</u>

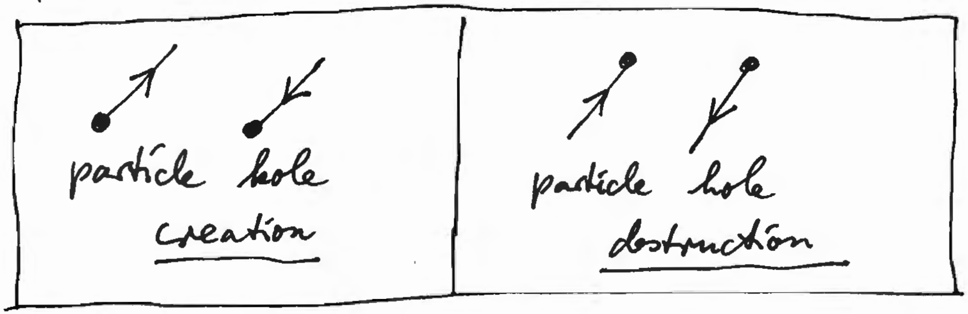
" " " <u>downward</u> arrow $\hat{=}$ <u>hole</u>

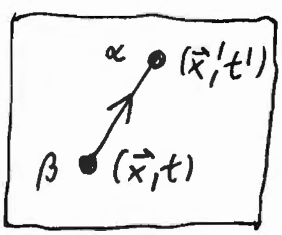
(b) 

<u>creation</u> process: solid line <u>emerging from</u> $(\vec{x},t)$

<u>destruction</u> process: solid line <u>terminating</u> at $(\vec{x},t)$

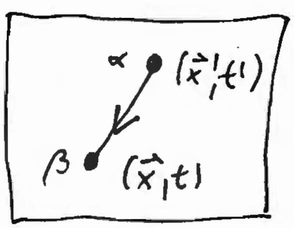
(c) from (a) and (b) we conclude:



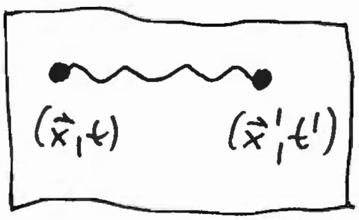
(d) 

$\hat{=}$ $\underline{i G_{\alpha\beta}(\vec{x}',t';\vec{x},t)}$ = <u>particle propagator (fermions)</u>

particle created at $(\vec{x},t)$, propagate to time t', where it is destroyed at $\vec{x}'$.

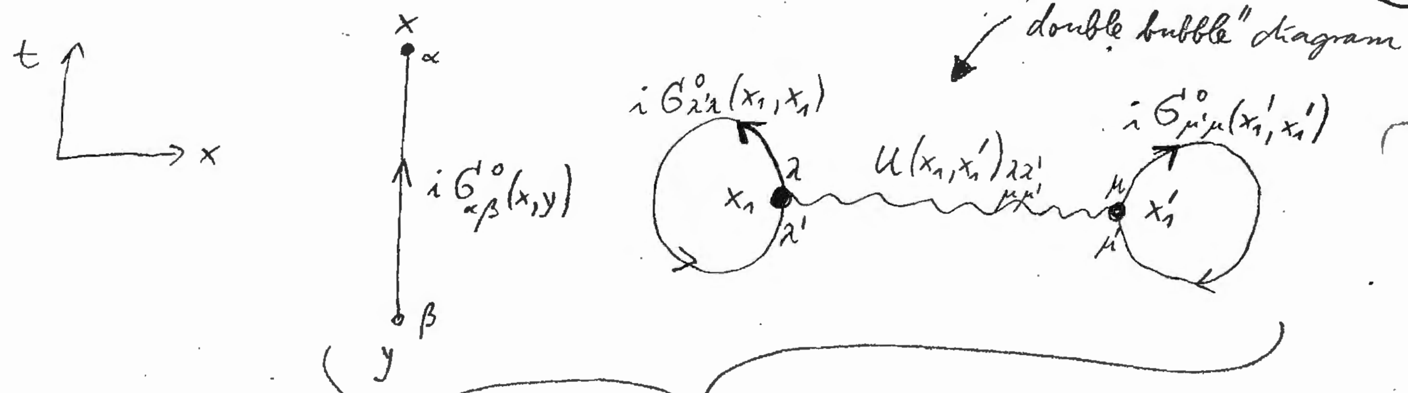


<u>fermion hole propagator</u>: hole is created at $(\vec{x},t)$, propagate to time t', where it is destroyed at $\vec{x}'$.

(e) 

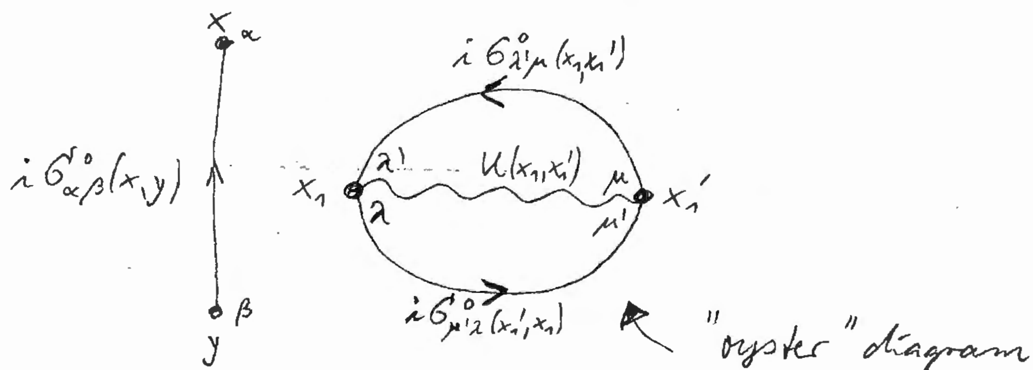
(instantaneous) <u>4-potential interaction</u>

$$U(x,x') = \delta(t-t')\,V(\vec{x},\vec{x}')$$

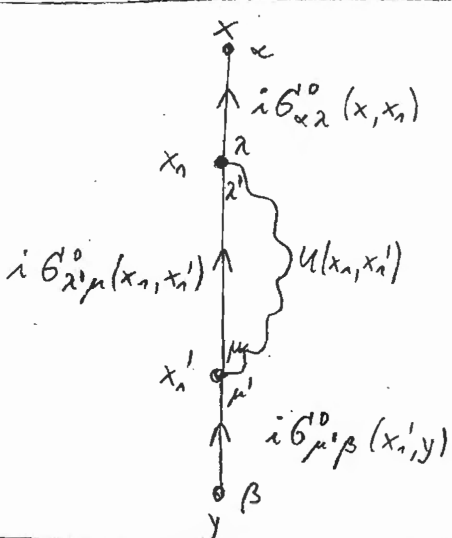


graph (A)

Similaly we find graphs (B) and (C):



graph (B)



graph (C)

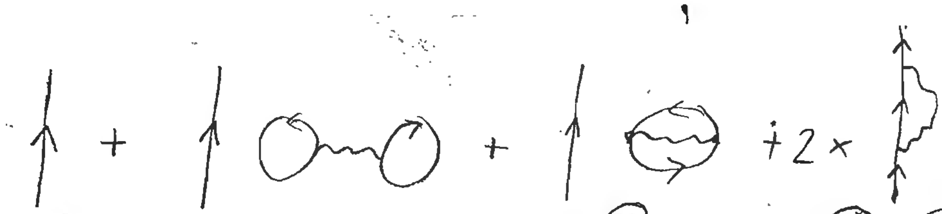
Graphs (D), (E), (F)

see FW, p. 93

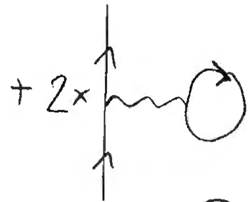
Note that graphs (A) and (B) are "disconnected" Feynman diagrams, whereas graph (C) is "connected".

We therefore find for the numerator of the propagator in 0th and 1st order pert. theory:

$i G^{0}_{\alpha\beta} + i \bar{G}^{(1)} =$



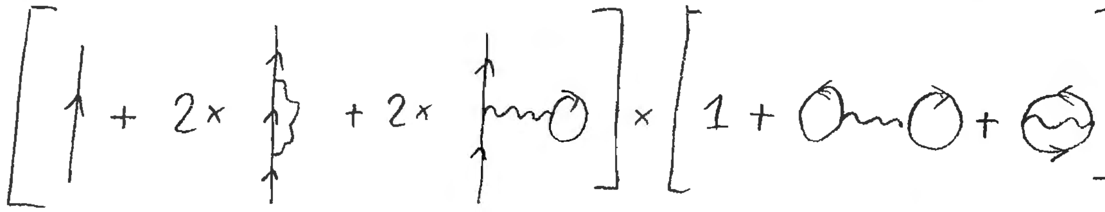
iG^0 (A) (B) (C)+(E)



(D), (F)

$=$ factorize, only terms with <u>one</u> wavy line are to be retained in first-order pert. theory $=$

$i G^{0}_{\alpha\beta} + i \bar{G}^{(1)} =$

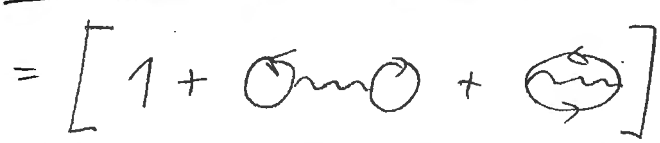


Let us now consider the denominator of $G_{\alpha\beta}$. From page 24 we find

$$\boxed{1 + i\mathcal{D}^{(1)}} = 1 + \frac{1}{2}\left(\frac{-i}{\hbar}\right) \sum_{\lambda\lambda'\mu\mu'} \int d^4x_1 \int d^4x_1' \, U(x_1, x_1')_{\lambda\lambda'\mu\mu'} \times$$

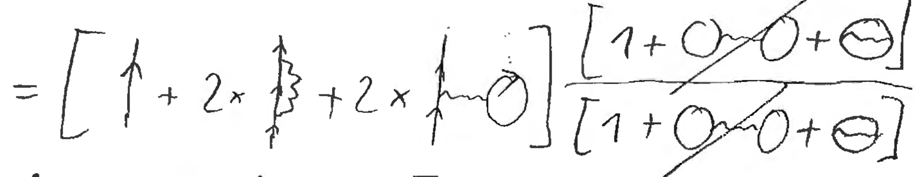
$$\times \langle \Phi_0 | T\{ \psi^{+}_{\lambda}(x_1) \psi^{+}_{\mu}(x_1') \psi_{\mu'}(x_1') \psi_{\lambda'}(x_1) \} | \Phi_0 \rangle$$

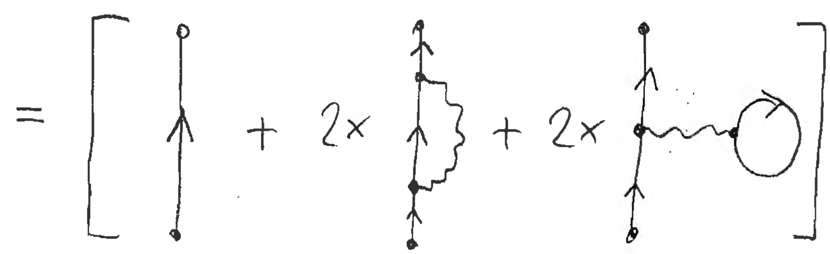
$=$ Wick's theorem, plot Feynman graphs $=$



Hence, in perturb. theory up to first-order we find for the propagator

$$\underline{\underline{i G_{\alpha\beta}}} = \frac{i G^{0}_{\alpha\beta} + i \bar{G}^{(1)}}{1 + i\mathcal{D}^{(1)}} =$$





Summary so far:

1st-order perturb. theory (n=1): 2 distinct Feynman diagrams involving
3 free propagators $G^{(0)}$
1 interaction line U

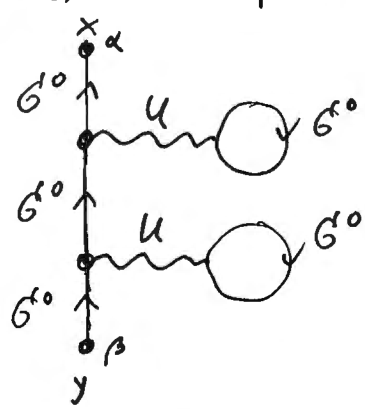
Note that these diagrams arise apparently from the connected part of the numerator, $i\tilde{G}^{(1)}$, of the propagator. Instead of the complicated expression for $G_{\alpha\beta}$ on p. 24, we may start from the much simpler expression

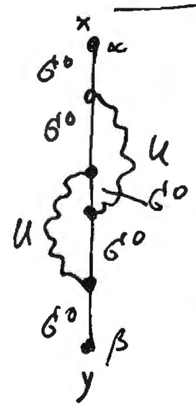
$$\boxed{i\,G_{\alpha\beta}(x,y) = \langle \Phi_0 | T\{\exp[-\tfrac{i}{\hbar}\int_{-\infty}^{+\infty} d\tau\, \hat{H}_1(\tau)]\,\hat{\psi}_\alpha(x)\,\hat{\psi}^+_\beta(y)\}|\Phi_0\rangle_{\text{connected}}}$$

FW, eq. (9.5)

which may be used with the expression for $\int_{-\infty}^{+\infty} d\tau\, \hat{H}_1(\tau)$ on p. 23 as the new starting point for perturbation expansion of the propagator.

In this way, one can show that there are 10 2nd-order distinct connected diagrams $G^{(2)}_{\alpha\beta}$ which are shown in FW, p. 100.
We show 2 of these as example:





We observe: 2nd-order (n=2)
5 free propagators $G^{(0)}$
2 interaction lines U

Generalize:

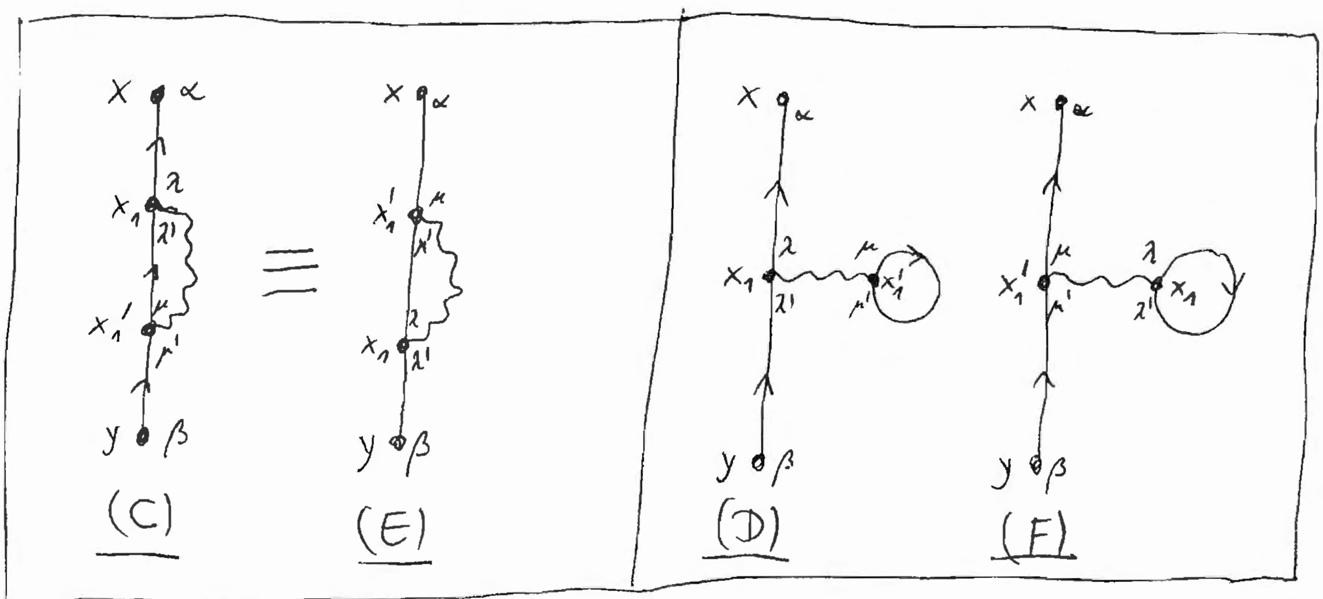
in n^{th}-order perturb. theory for $G^{(n)}$:
n interaction lines U
$(2n+1)$ free propagators $G^{(0)}$

Rules for Feynman diagrams of $G_{\alpha\beta}(x,y)$ in coord. space

Using Wick's theorem, one can study not only 1st, but 2nd-order, 3rd-order pert. theory etc. By examining the results in pictorial representation, one can establish the foll. Feynman rules for $G_{\alpha\beta}$ in n-th order pert. theory [note that one can always derive the diagrams using Wick's theorem, but using the heuristic Feynman rules is certainly quicker and easier]:

(a) In n-th order pert. theory, each Feynman diagram of $G_{\alpha\beta}$ contains n interaction lines (∿∿∿) and $(2n+1)$ free propagators $G^0_{\alpha\beta}$. Draw only connected and (topologically) distinct diagrams.

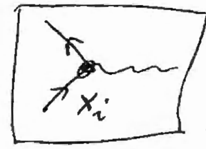
Counterexample: The diagrams (C) and (E) in FW p. 93 are topologically equivalent, and so are the diagrams (D) and (F)



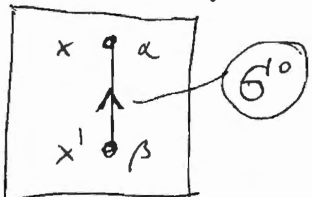
because they differ only by relabeling of dummy integration variables $x_1 \leftrightarrow x_1'$ and summation over dummy spin indices!

|| In the rules below, such terms will be drawn only once, and the factor 2 is included in the rules given! ||

(b) Label each vertex with a 4-dim. space-time point x_i :



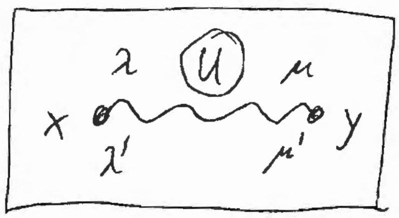
(c) A free propagator $G^0_{\alpha\beta}(x,x')$ is represented by a solid line emerging from point x' and ending in point x. Draw an arrow from x' to x.



(d) The instantaneous (nonrel) 2-body int. potential

$$U(x,y)_{\lambda\lambda',\mu\mu'} = V(\vec{x},\vec{y})_{\lambda\lambda',\mu\mu'}\,\delta(t_x - t_y)$$

is represented by a wavy line. The spin indices (λ,λ') refer to the first coord. x, the indices (μ,μ') to the second coord. y :



(e) Integrate over all internal space-time points $x_i = (\vec{x}_i, t_i)$, but leave out the two external endpoints x and y of the propagator $G'_{\alpha\beta}(x,y)$.

(f) Sum over all internal spin indices along each continuous fermion line.

(g) If F is the number of closed fermion loops in the diagram, assign a factor $(-1)^F$.

(h) To compute $G'^{(n)}_{\alpha\beta}(x,y)$, assign a factor $\left(\frac{i}{\hbar}\right)^n$ to each n-th order Feynman diagram.

(i) For propagators with equal time variables:

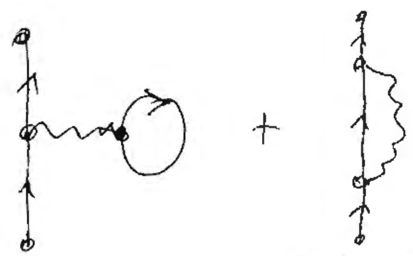
$$G^{0}_{\alpha\beta}(\vec{x},t;\vec{x}',t) \rightarrow \lim_{\varepsilon\to 0} G^{0}_{\alpha\beta}(\vec{x},t;\vec{x}',t+\varepsilon)$$

$\boxed{\text{Application}}$

From rule (a) we find that in <u>0th-order pert. theory</u>, there are <u>0 int. lines</u> and <u>1 free propagator</u>, i.e. we obtain the trivial result $G^{0}_{\alpha\beta}(x,y)$.

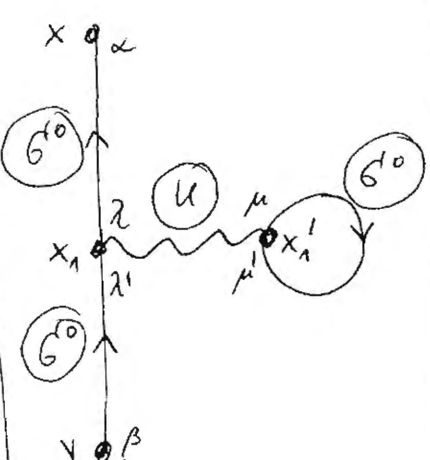
In <u>1st-order pert. theory</u>, we are supposed to draw <u>all connected and</u> topolog. <u>distinct</u> diagrams with <u>1 int. line</u> and <u>3 free prop.</u> $\Rightarrow$

2 possibilities:



$\hat{=}$ <u>(D+F)</u> $\qquad$ + $\qquad$ $\hat{=}$ <u>(C+E)</u>

Let us calculate the left diagram (D+F) according to the above rules. Assign arbitrary internal coord. + spin indices



$$\Rightarrow G^{(1)\,(D+F)}_{\alpha\beta}(x,y) =$$

$$= \underbrace{\left(\frac{i}{\hbar}\right)}_{\substack{\text{rule (h),}\\ \text{1st-order}}} \times \underbrace{(-1)}_{\substack{\text{rule (g)}\\ (F=1)\\ \text{one closed loop}}} \underbrace{\int d^4x_1 \int d^4x_1'}_{\substack{\text{rule (e)}\\ \text{(internal variables)}}} \times$$

$$\times \underbrace{\sum_{\lambda\lambda'\mu\mu'}}_{\text{rule (f)}} G^{0}_{\lambda'\beta}(x_1,y)\, G^{0}_{\alpha\lambda}(x,x_1)\, U(x_1,x_1')_{\lambda\lambda'\mu\mu'}$$

$$\times\, G^{0}_{\mu'\mu}(x_1',x_1')$$

Note: this result agrees with Wick theorem for (D) + (F) in FW eq. (9.1) on p. 92.

The propagator $G_{\alpha\beta}$ and relation to observables

Using the perturbation expansion of $G_{\alpha\beta}$

$$\boxed{G_{\alpha\beta} = G^{(0)}_{\alpha\beta} + G^{(1)}_{\alpha\beta} + G^{(2)}_{\alpha\beta} + \dots}$$

it is possible to compute the following observables

1) g.s. expectation value of any one-body operator
2) g.s. energy of the system
3) energy spectrum of excited states

Details

1 one-body operators in g.s.

Ref: Fetter & Walecka, p. 66–67

Consider an arbitrary one-body operator $J = \sum_{i=1}^{N} J(\vec{x}_i)$ in coord. representation. In occupation # space we obtain (lecture notes ch. 2, p. 27):

$$\underbrace{\hat{J}}_{\substack{\text{occup.\#}\\ \text{space}}} = \int d^3x \; \underbrace{\hat{\psi}^+(\vec{x})}_{\text{field op.}} \; \underbrace{J(\vec{x})}_{\substack{\text{s.p.}\\ \text{operator}}} \; \underbrace{\hat{\psi}(\vec{x})}_{\text{field op.}}$$

For $s=\frac{1}{2}$ fermions, the field operators are column/row vectors and $J(x)$ is, in general, a matrix in spin-space, i.e.

$$\hat{J} = \int d^3x \; \underbrace{\left(\hat{\psi}_1^*(\vec{x}), \hat{\psi}_2^*(\vec{x})\right) \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \begin{pmatrix} \hat{\psi}_1(\vec{x}) \\ \hat{\psi}_2(\vec{x}) \end{pmatrix}}_{=\text{Scalar}} \qquad (*)$$

Let's write out (*) in components:

$$\hat{J} = \int d^3x \underbrace{\sum_{\beta,\alpha=1}^{2} \hat{\psi}_\beta^+(\vec{x})\, J_{\beta\alpha}(\vec{x})\, \hat{\psi}_\alpha(\vec{x})}_{\stackrel{\text{def}}{=} \hat{j}(\vec{x})}$$

which can be recast in the form

$$\boxed{\begin{aligned} &\hat{J} = \int d^3x\, \hat{j}(\vec{x}) && \text{one-body op.} \\ &\hat{j}(\vec{x}) = \sum_{\beta,\alpha=1}^{2} \hat{\psi}_\beta^+(\vec{x})\, J_{\beta\alpha}(\vec{x})\, \hat{\psi}_\alpha(\vec{x}) && \text{one-body op. density} \end{aligned}} \quad \underline{(1)}$$

Let's consider <u>3 examples:</u>

<u>a) number density $\rho(\vec{x})$</u>

In chapter 2, p. 30 we found the density operator in 2nd quant.

$$\hat{\rho}(\vec{x}) = \hat{\psi}^+(\vec{x})\psi(\vec{x}) = \sum_{\beta=1}^{2} \hat{\psi}_\beta^+(\vec{x})\, \hat{\psi}_\beta(\vec{x}) \stackrel{(1)}{=}$$

$$= \sum_{\alpha,\beta=1}^{2} \hat{\psi}_\beta^+(\vec{x})\, \rho_{\beta\alpha}(\vec{x})\, \hat{\psi}_\alpha(\vec{x}) \implies \boxed{\rho_{\beta\alpha}(\vec{x}) = \delta_{\alpha\beta}} \quad \underline{(2)}$$

<u>b) spin density $\vec{\sigma}(\vec{x})\cdot\frac{\hbar}{2}$</u>

The total spin is a one-body operator: $\vec{S} = \sum_{i=1}^{N} \frac{\hbar}{2}\vec{\sigma}(i)$

Pauli matrix

$$\leadsto \boxed{J_{\beta\alpha}(\vec{x}) = \frac{\hbar}{2}\vec{\sigma}_{\beta\alpha}} \quad \underline{(3)}$$

Pauli matrix components

<u>c) kinetic energy density</u>

$$\boxed{J_{\beta\alpha}(\vec{x}) \to \tau_{\beta\alpha}(\vec{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2\right)\delta_{\alpha\beta}} \quad \underline{(4)}$$

We now compute the g.s. expectation value of the operator density

$$\langle \hat{j}(\vec{x})\rangle_{g.s.} \stackrel{def}{=} \frac{\langle\Psi_0|\hat{j}(\vec{x})|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \stackrel{(1)}{=}$$

$$= \sum_{\beta,\alpha=1}^{2} \frac{\langle\Psi_0|\hat{\psi}_\beta^+(\vec{x})\, J_{\beta\alpha}(\vec{x})\, \psi_\alpha(\vec{x})|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} =$$

$$= \underline{\underline{\lim_{\vec{x}'\to\vec{x}}}} \sum_{\beta,\alpha=1}^{2} \underline{\underline{J_{\beta\alpha}(\vec{x})}} \frac{\langle\Psi_0|\hat{\psi}_\beta^+(\underline{\underline{\vec{x}'}})\,\hat{\psi}_\alpha(\vec{x})|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \Bigg\} \begin{matrix}\text{everything}\\ \text{in}\\ \underline{\text{S-picture!}}\end{matrix}$$

The underlined term looks very much like the propagator. Indeed, in the S-picture, one finds [FW (7.6)]:

$$\text{for } t'>t: \quad i\, G_{\alpha\beta}(\vec{x},t;x't') = -e^{iE(t'-t)/\hbar} \times \frac{\langle\Psi_0|\hat{\psi}_\beta^+(\vec{x}')\, e^{i\hat{H}(t-t')/\hbar}\,\hat{\psi}_\alpha(x)|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}$$

Consider this at time $t'=t+\varepsilon$ ($\varepsilon>0$, small):

$$i\, G_{\alpha\beta}(\vec{x},t;\vec{x}',t+\varepsilon) = -\, e^{iE\varepsilon/\hbar}\, \frac{\langle\Psi_0|\hat{\psi}_\beta^+(\vec{x}')\, e^{-i\hat{H}\varepsilon/\hbar}\, \psi_\alpha(\vec{x})|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} \qquad (5)$$

hence in the limit $\varepsilon\to 0$:

$$\boxed{\lim_{\varepsilon\to 0} i\, G_{\alpha\beta}(\vec{x},t;\vec{x}',t+\varepsilon) = -\, \frac{\langle\Psi_0|\hat{\psi}_\beta^+(\vec{x}')\,\hat{\psi}_\alpha(\vec{x})|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}} \qquad (6)$$

Insert (6) into above expression for $\langle\hat{j}(\vec{x})\rangle_{g.s.}$:

$$\boxed{\langle j(\vec{x})\rangle_{g.s.} = \lim_{\vec{x}'\to\vec{x}} \sum_{\beta,\alpha=1}^{2} J_{\beta\alpha}(\vec{x}) \lim_{\varepsilon\to 0} (-i)\, G_{\alpha\beta}(\vec{x},t;\vec{x}',t+\varepsilon)} \qquad (7)$$

This is our desired result for the one-body op. density.

Applications

a) numbers density $\langle \rho(\vec{x}) \rangle_{g.s.}$

From eq. (2): $\mathcal{J}_{\beta\alpha}(\vec{x}) \to \rho_{\beta\alpha}(\vec{x}) = \delta_{\alpha\beta}$

From eq. (7): $\langle \rho(\vec{x}) \rangle_{g.s.} = \lim_{\vec{x}' \to \vec{x}} \sum_{\beta,\alpha=1}^{2} \delta_{\alpha\beta} \lim_{\varepsilon\to 0} (-i)\, G_{\alpha\beta}(\vec{x},t;\vec{x}',t+\varepsilon)$

$$= (-i) \lim_{\varepsilon \to 0} \underbrace{\sum_{\alpha=1}^{2} G_{\alpha\alpha}(\vec{x},t;\vec{x},t+\varepsilon)}_{=\text{trace } G}$$

$$\Rightarrow \boxed{\langle \rho(\vec{x}) \rangle_{g.s.} = (-i) \lim_{\varepsilon\to 0} \text{trace } G(\vec{x},t;\vec{x},t+\varepsilon)}$$ g.s. number density

b) spin density $\langle \vec{\sigma}(\vec{x}) \rangle_{g.s.} \cdot \frac{\hbar}{2}$

From eq. (3): $\mathcal{J}_{\beta\alpha}(\vec{x}) \to \frac{\hbar}{2} \vec{\sigma}_{\beta\alpha}$

$$\frac{\hbar}{2} \langle \vec{\sigma}(\vec{x}) \rangle_{g.s.} \overset{(7)}{=} \lim_{\vec{x}' \to \vec{x}} \sum_{\alpha,\beta=1}^{2} \left(\frac{\hbar}{2}\right) \vec{\sigma}_{\beta\alpha} \lim_{\varepsilon\to 0} (-i)\, G_{\alpha\beta}(\vec{x},t;\vec{x}',t+\varepsilon)$$

$$\boxed{\frac{\hbar}{2} \langle \vec{\sigma}(\vec{x}) \rangle_{g.s.} = (-i) \frac{\hbar}{2} \lim_{\varepsilon\to 0} \text{trace}\left[\vec{\sigma} \cdot G(\vec{x},t;\vec{x},t+\varepsilon\right]}$$ spin density

c) kinetic energy density $\langle \tau(\vec{x}) \rangle_{g.s.}$

From eq. (4): $\mathcal{J}_{\beta\alpha}(\vec{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2\right)\delta_{\alpha\beta}$

$$\langle \tau(\vec{x}) \rangle_{g.s.} = \lim_{\vec{x}' \to \vec{x}} \sum_{\beta,\alpha=1}^{2} \left(-\frac{\hbar^2}{2m}\nabla^2\right) \delta_{\alpha\beta} \lim_{\varepsilon\to 0} (-i)\, G_{\alpha\beta}(\vec{x},t,\vec{x}',t+\varepsilon)$$

$$= (-i) \lim_{\vec{x}' \to \vec{x}} \lim_{\varepsilon \to 0} \sum_{\alpha=1}^{2} \left(-\frac{\hbar^2}{2m}\nabla^2\right) G_{\alpha\alpha}(\vec{x},t,\vec{x}',t+\varepsilon)$$

$$\Rightarrow \boxed{\langle \tau(\vec{x}) \rangle_{g.s.} = (-i) \lim_{\vec{x}' \to \vec{x}} \lim_{\varepsilon \to 0} \left(-\frac{\hbar^2}{2m}\nabla^2\right) \text{trace}\, G(\vec{x},t;\vec{x}',t+\varepsilon)}$$

g.s. kin. energy density

Other observables from propagator $G_{\alpha\beta}$

- g.s. energy of the system

Ref: Fetter & Walecka, p. 67 – 68

$$\boxed{E = -\frac{1}{2} i \int d^3x \lim_{\vec{x}' \to \vec{x}} \lim_{\varepsilon \to 0} \left[i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\nabla^2 \right] \text{trace}\, G(\vec{x},t;\vec{x}',t+\varepsilon)}$$

$\uparrow$ exact

Note that the propagator contains info about 2-body int. potential $V^{(2)}(\vec{x},\vec{x}')_{\alpha\alpha'\beta\beta'}$.

Note: there is an alternative expression for $\Delta E = E - E_0$, based on Gell-Mann & Low theorem, see next page!

- energy spectrum of excited states: "Lehmann representation"

Ref: Fetter & Walecka, p. 72-79

If Hamiltonian is translationally invariant, i.e. $[\hat{H}, \hat{\vec{P}}] = 0$, one considers the Fourier transform of G:

$$G_{\alpha\beta}(\vec{k}, \omega).$$

One can show that for given momentum $\hbar\vec{k}$, the propagator in momentum space $G_{\alpha\beta}(\vec{k},\omega)$ has simple poles at the exact excitation energies $\hbar\omega$.

Ground state energy shift in perturb. theory — Goldstone's theorem

We start from the Gell-Mann and Low theorem for the g.s. energy shift, see lecturenotes, p. 19

notation: $\hat{H}_0 \to E_0$, $\hat{H} = \hat{H}_0 + \hat{H}_1 \to E$ exact

$$\Delta E = E - E_0 = \lim_{\varepsilon \to 0} \frac{\langle \phi_0 | \hat{H}_1 \hat{U}_\varepsilon^{(I)}(0,-\infty) | \phi_0 \rangle}{\langle \phi_0 | \hat{U}_\varepsilon^{(I)}(0,-\infty) | \phi_0 \rangle}$$

Inserting $\hat{U}_\varepsilon^{(I)}(t,t_0)$ and taking $\lim_{\varepsilon \to 0}$, we obtain (notes p. 17):

$$\Delta E = E - E_0 = \frac{\langle \phi_0 | \hat{H}_1 T\{\exp[-\frac{i}{\hbar}\int_{-\infty}^{0} dt\, \hat{H}_1^{(I)}(t)]\} | \phi_0 \rangle}{\langle \phi_0 | T\{\exp[-\frac{i}{\hbar}\int_{-\infty}^{0} dt\, \hat{H}_1^{(I)}(t)]\} | \phi_0 \rangle} = \frac{N}{D}$$

Observe now that

$$\underline{\hat{H}_1} \equiv \hat{H}_1^{(S)} = \underline{\hat{H}_1^{(I)}(t=0)} .$$

Therefore, we can incorporate this term into the T-product of the numerator N, because $\hat{H}_1^{(I)}(0)$ corresponds to a later time than all the other factors in the Taylor series expansion of $\exp[-\frac{i}{\hbar}\cdots]$, because the terms

$$\hat{H}_1^{(I)}(t_1)\, \hat{H}_1^{(I)}(t_2) \cdots \hat{H}_1^{(I)}(t_n)$$

occur in the time interval $[-\infty, 0]$. Hence, the numerator can be written in the form

$$N = \langle \phi_0 | T\{ \hat{H}_1^{(I)}(0) \cdot \exp[-\frac{i}{\hbar}\int_{-\infty}^{0} dt\, \hat{H}_1^{(I)}(t)]\} | \phi_0 \rangle .$$

One now proceeds in a similar way as in the calculation of the propagator $G_{\alpha\beta}$: the denominator D cancels against the disconnected parts of the numerator N_1 and one obtains

$$\boxed{\Delta E = E - E_0 = \langle \phi_0 | T\{ \hat{H}_1^{(I)}(0) \cdot \exp[-\tfrac{i}{\hbar}\int_{-\infty}^{0} dt\, \hat{H}_1^{(I)}(t)]\} | \phi_0 \rangle_{\text{connected}} \quad (*)}$$

FW (9.53)

This is the time-dependent version of Goldstone's theorem (1957) for the g.s. energy shift in perturb. theory.

It turns out that one can carry out the time-integrations explicitly in which case one arrives at the time-independent version of Goldstone's theorem:

$$\boxed{\Delta E = E - E_0 = \langle \phi_0 | \hat{H}_1 \sum_{n=0}^{\infty} \left(\frac{1}{E_0 - \hat{H}_0} \hat{H}_1 \right)^n | \phi_0 \rangle_{\text{connected}}}$$

FW (9.48)

We will now use the time-dependent version based on the Feynman-Dyson time-ordered product, which "has the fundamental advantage ~~that~~ of combining many terms of time-indep. perturb. theory into a single Feynman diagram" (FW, p. 115).

Expand the exponential in eq. (*) above:

$$\Delta E = E - E_0 = \Delta E^{(1)} + \Delta E^{(2)} + \ldots$$

with

$$\Delta E^{(1)} = \langle \phi_0 | T\{ \hat{H}_1^{(I)}(0) \} | \phi_0 \rangle_{\text{connected}}$$

$$\Delta E^{(2)} = \langle \phi_0 | T\{ \hat{H}_1^{(I)}(0) \cdot (-\tfrac{i}{\hbar}) \int_{-\infty}^{0} dt\, \hat{H}_1^{(I)}(t) \} | \phi_0 \rangle_{\text{connected}}$$

time-dep. version of Goldstone's theorem

energy shift $\Delta E^{(1)}$ in first-order P.T.

$$\Delta E^{(1)} = \langle \phi_0 | T\{\hat{H}_1^{(I)}(\tau=0)\} | \phi_0 \rangle_{\text{connected}}$$

Insert interaction Hamiltonian in I-picture at time $\tau = 0$; from p. 22, eq. (2) we obtain

$$\left. \begin{aligned} \hat{H}_1^{(I)}(\tau=0) = \frac{1}{2} \sum_{\lambda\lambda'\mu\mu'} \int d^3x \int d^3x' \, \hat{\psi}_\lambda^{(I)+}(\vec{x},0)\, \hat{\psi}_\mu^{(I)+}(\vec{x}',0) \times \\ \times V(\vec{x},\vec{x}')_{\lambda\lambda'\mu\mu'} \, \hat{\psi}_{\mu'}^{(I)}(\vec{x}',0)\, \hat{\psi}_{\lambda'}^{(I)}(\vec{x},0) \end{aligned} \right\}$$

We simplify the notation now and leave out the (I)-superscript and the common time argument $(\tau = 0)$:

$$\left. \begin{aligned} \Delta E^{(1)} = \frac{1}{2} \sum_{\lambda\lambda'\mu\mu'} \int d^3x \int d^3x' \, V(\vec{x},\vec{x}')_{\lambda\lambda'\mu\mu'} \times \\ \times \langle \phi_0 | T\{\hat{\psi}_\lambda^+(\vec{x})\, \hat{\psi}_\mu^+(\vec{x}')\, \hat{\psi}_{\mu'}(\vec{x}')\, \hat{\psi}_{\lambda'}(\vec{x})\} | \phi_0 \rangle_{\text{connected}} \end{aligned} \right\} \quad (1)$$

Evaluate T-product using Wick's theorem. Only full contractions contribute. The only non-zero contractions are between $\hat{\psi}$ and $\hat{\psi}^+$; there are 2 contributing terms:

$$\langle \phi_0 | T\{\cdots\} | \phi_0 \rangle_{\text{conn}} =$$

$$\langle \phi_0 | N\{\hat{\psi}_\lambda^+(\vec{x})\, \hat{\psi}_\mu^+(\vec{x}')\, \hat{\psi}_{\mu'}(\vec{x}')\, \hat{\psi}_{\lambda'}(\vec{x})\} | \phi_0 \rangle_{\text{conn}}$$

$$+ \langle \phi_0 | N\{\hat{\psi}_\lambda^+(\vec{x})\, \hat{\psi}_\mu^+(\vec{x}')\, \hat{\psi}_{\mu'}(\vec{x}')\, \hat{\psi}_{\lambda'}(\vec{x})\} | \phi_0 \rangle_{\text{conn}} =$$

$$= \Big[\ \hat{\psi}_\mu^+(\vec{x}')\, \hat{\psi}_{\mu'}(\vec{x}') \cdot \hat{\psi}_\lambda^+(\vec{x})\, \hat{\psi}_{\lambda'}(\vec{x}) +$$

$$(-1)\, \underbrace{\hat\psi^+_{\lambda}(\vec x)\, \hat\psi_{\mu'}(\vec x\,')}\cdot \hat\psi^+_{\mu}(\vec x\,')\underbrace{\hat\psi_{\lambda'}(\vec x)}\Big]_{conn}$$

where the $(-)$ sign in the 2nd term is because of the rearrangement of one fermion field operator inside the N-product. Using (p. 26)

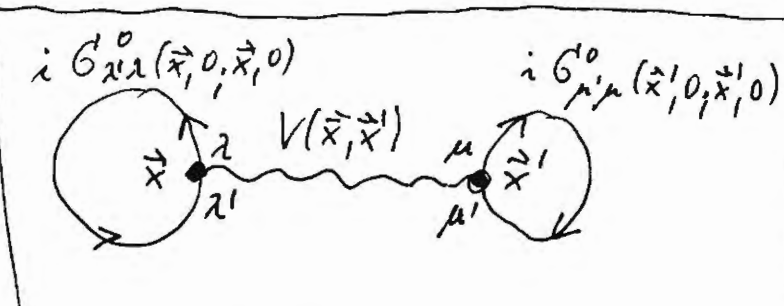
$$\underbrace{\hat\psi^+\hat\psi} = -\underbrace{\hat\psi\,\hat\psi^+} = -(i\,G^0)$$

we obtain:

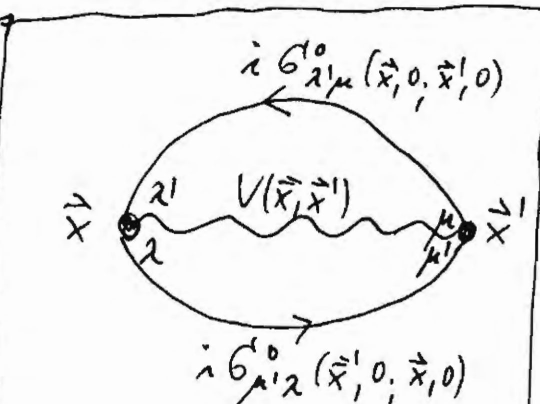
$$\left.\begin{aligned}\langle\phi_0|T\{\dots\}|\phi_0\rangle = \Big[(-i)G^0_{\mu'\mu}(\vec x\,',\vec x\,')\,(-i)\,G^0_{\lambda'\lambda}(\vec x,\vec x) - \\ -(-i)\,G^0_{\mu'\lambda}(\vec x\,',\vec x)\,(-i)\,G^0_{\lambda'\mu}(\vec x,\vec x\,')\Big]_{conn}\end{aligned}\right\} \quad \underline{(2)}$$

Inserting eq. (2) into eq. (1) on p. 42, and adding the time argument $(\tau=0)$ in the Green's functions (which we had left out for notational convenience:

$$\boxed{\begin{aligned}\Delta E^{(1)} = \frac{1}{2}\sum_{\lambda\lambda'\mu\mu'}\int d^3x\int d^3x'\,V(\vec x,\vec x\,')_{\lambda\lambda'\mu\mu'}\,\times \\ \times\Big[\underbrace{i\,G^0_{\mu'\mu}(\vec x\,',0;\vec x\,',0)\; i\,G^0_{\lambda'\lambda}(\vec x,0;\vec x,0)}_{\text{graph A}} \\ \underbrace{-i\,G^0_{\mu'\lambda}(\vec x\,',0;\vec x,0)\; i\,G^0_{\lambda'\mu}(\vec x,0;\vec x\,',0)}_{\text{graph B}}\Big]_{\text{connected}}\end{aligned}} \quad \underline{(3)}$$



graph A



graph B

As we can see, <u>both graphs are connected</u> and hence contribute to $\Delta E^{(1)}$. In order to evaluate eq. (3) further, we need the expressions for the <u>free propagator iG^0 at equal times $t' = t = 0$.</u> According to the Feynman rules stated on p. 34, we are to replace

$$G^0_{\alpha\beta}(\vec{x},t;\vec{x}',t) \to \lim_{\varepsilon\to 0} G^0_{\alpha\beta}(\vec{x},t;\vec{x}',\underbrace{t+\varepsilon}_{=t'}) \qquad (\varepsilon > 0)$$

Using the expression on p. 14 for $G^0_{\alpha\beta}$ (in the <u>discrete form</u> involving $\sum_{\vec{k}}\dots$), we obtain

$$\left.\begin{aligned} iG^0_{\alpha\beta}(\vec{x},t;\vec{x}',t+\varepsilon) &= \frac{\delta_{\alpha\beta}}{L^3}\sum_{\vec{k}} e^{i[\vec{k}\cdot(\vec{x}-\vec{x}')-\omega_k(-\varepsilon)]}\times \\ &\times\left[-\theta(\varepsilon)\,\theta(k_F-k)\right] \end{aligned}\right\}$$

In the <u>limit $\varepsilon \to 0$</u>, we find (at $t=0$):

$$iG^0_{\alpha\beta}(\vec{x},0;\vec{x}',0) = -\frac{\delta_{\alpha\beta}}{L^3}\sum_{\vec{k}<k_F} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

Using the plane-wave spinor WF's (eigenfunctions of free Hamiltonian $\hat{H}_0 = \hat{T}$)

$$\boxed{\varphi_{k\alpha}(\vec{x}) = L^{-3/2}\, e^{i\vec{k}\cdot\vec{x}}\,\eta_\alpha} \qquad \underline{(4)}$$

we find

$$\boxed{iG^0_{\alpha\beta}(\vec{x},0;\vec{x}',0) = -\delta_{\alpha\beta}\underbrace{\sum_{\vec{k}<k_F}\varphi^+_{\vec{k}\alpha}(\vec{x}')\,\varphi_{\vec{k}\alpha}(\vec{x})}_{=\rho_{\alpha\alpha}(\vec{x}',\vec{x})\ [\text{ch. 6, p. 7}]} = -\delta_{\alpha\beta}\,\rho^{(0)}_{\alpha\alpha}(\vec{x}',\vec{x})}$$

i.e. we have proven that the <u>free propagator iG^0 at equal times</u> equals the <u>1-body density matrix $\rho^{(0)}_{\alpha\alpha}(\vec{x}',\vec{x})$.</u>

Inserting (4) into eq. (3) on p. 43 and observing that the diagonal elements of the density matrix are the densities, i.e.

$$i\, G^{0}_{\alpha\beta}(\vec{x},0;\vec{x},0) = -\delta_{\alpha\beta}\, \rho^{(0)}_{\alpha\alpha}(\vec{x},\vec{x}) \equiv -\delta_{\alpha\beta}\, \rho^{(0)}_{\alpha}(\vec{x})$$

we obtain

$$\left.\begin{aligned}\Delta E^{(1)} = \frac{1}{2} \sum_{\lambda\lambda'\mu\mu'} \int d^3x \int d^3x'\, V(\vec{x},\vec{x}')_{\lambda\lambda'\mu\mu'} \times \\ \Big[\left(-\delta_{\mu'\mu}\, \rho^{(0)}_{\mu}(\vec{x}')\right)\left(-\delta_{\lambda'\lambda}\, \rho^{(0)}_{\lambda}(\vec{x})\right) - \left(-\delta_{\mu'\lambda}\, \rho^{(0)}_{\lambda\lambda}(\vec{x}',\vec{x})\right) \cdot \\ \cdot \left(-\delta_{\lambda'\mu}\, \rho^{(0)}_{\mu\mu}(\vec{x},\vec{x}')\right)\Big]\end{aligned}\right\}$$

Because of the double Kronecker delta's, the quadruple sum reduces to a double sum:

$$\boxed{\begin{aligned}\Delta E^{(1)} = \frac{1}{2} \sum_{\lambda\mu} \int d^3x \int d^3x' \Big[& \underbrace{V(\vec{x},\vec{x}')_{\lambda\lambda,\mu\mu}\, \rho^{(0)}_{\mu}(\vec{x}')\, \rho^{(0)}_{\lambda}(\vec{x})}_{\text{graph A = direct term}} \\ & \underbrace{- V(\vec{x},\vec{x}')_{\lambda\mu,\mu\lambda}\, \rho^{(0)}_{\mu\mu}(\vec{x},\vec{x}')\, \rho^{(0)}_{\lambda\lambda}(\vec{x}',\vec{x})}_{\text{graph B} \,\hat{=}\, \text{exchange term}} \Big]\end{aligned}} \qquad (5)$$

We observe now that graph A depends on the product of the densities at points $\vec{x}$ and $\vec{x}'$, i.e. it is a "direct energy" term, whereas graph B enters with opposite sign and involves a product of density matrices ("exchange energy" term).

We can simplify this expression further if we assume that the 2-body interaction is spin-independent:

$$\boxed{V(\vec{x},\vec{x}')_{\lambda\lambda'\mu\mu'} \longrightarrow V(\vec{x},\vec{x}')} \qquad (6a)$$

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$$\left.\begin{aligned}\Delta E^{(1)} = \frac{1}{2}\int d^3x \int d^3x' \, V(\vec{x},\vec{x}')\Big[\sum_\lambda \rho_\lambda^{(0)}(\vec{x}) \sum_\mu \rho_\mu^{(0)}(\vec{x}') - \\ - \sum_\mu \rho_{\mu\mu}^{(0)}(\vec{x},\vec{x}') \sum_\lambda \rho_{\lambda\lambda}^{(0)}(\vec{x}',\vec{x}) \Big]\end{aligned}\right\} \quad \text{(6b)}$$

$$\Rightarrow \Delta E^{(1)} = \frac{1}{2}\int d^3x \int d^3x' \, V(\vec{x},\vec{x}')\Big[\rho^{(0)}(\vec{x})\rho^{(0)}(\vec{x}') - \rho^{(0)}(\vec{x},\vec{x}')\rho^{(0)}(\vec{x}',\vec{x}) \Big]$$

or finally

fermion pair correlation function

$$\boxed{\Delta E^{(1)} = \frac{1}{2}\int d^3x \int d^3x' \, V(\vec{x},\vec{x}')\Big[\underbrace{\rho^{(0)}(\vec{x})\rho^{(0)}(\vec{x}')}_{\text{"direct" term}} - \underbrace{|\rho^{(0)}(\vec{x},\vec{x}')|^2}_{\text{"exchange" term}} \Big]} \quad \text{(6c)}$$

Note that the structure of this energy is the same as in the HF theory of atoms (see chapter 7, p. 30); however, there is one fundamental difference: the HF-theory yields realistic densities, whereas here in 1st-order perturb. theory the densities are associated with the plane-wave spinor WF's of eq. (4), i.e. we have

$$\rho^{(0)}(\vec{x}) \equiv \rho_0 = \text{homogeneous density}$$

and the exchange term of the pair correlation function becomes (see chapter 6, p. 9):

$$|\rho^{(0)}(\vec{x},\vec{x}')|^2 = \left(3\rho_0 \frac{j_1(k_F|\vec{x}-\vec{x}'|)}{k_F|\vec{x}-\vec{x}'|} \right)^2$$

resulting in

$$\Delta E^{(1)} = \frac{\rho_0^2}{2}\int d^3x \int d^3x' \, V(|\vec{x}-\vec{x}'|)\left[1 - \left(\frac{3 j_1(k_F|\vec{x}-\vec{x}'|)}{k_F|\vec{x}-\vec{x}'|} \right)^2 \right]$$

Define the variable substitution

$$z \stackrel{\text{def}}{=} |\vec{x}-\vec{x}'|$$

we obtain

$$\Delta E^{(1)} = \frac{\rho_0^{\,2}}{2} \int d^3x \int d^3x'\, V(z) \left[1 - \left(\frac{3\, j_1(k_F z)}{k_F z} \right)^2 \right]$$

Using the homogeneity of the system (constant density ρ_0) in 1st-order P.T., we can shift the origin of the integration

$$\int d^3x' \rightarrow \int d^3|\vec{x}-\vec{x}'| = \int d^3z$$

and we arrive at

$$\Delta E^{(1)} = \frac{\rho_0^{\,2}}{2} \underbrace{\int d^3x}_{=L^3} \int d^3z\, V(z) \left[1 - \left(\frac{3\, j_1(k_F z)}{k_F z} \right)^2 \right]$$

Finally, we obtain an expression for the energy density change in 1st-order perturb. theory:

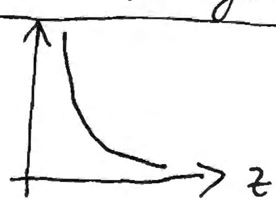
$$\boxed{\frac{\Delta E^{(1)}}{L^3} = \frac{\rho_0^{\,2}}{2} \int d^3z\, V(z) \left[1 - \left(\frac{3\, j_1(k_F z)}{k_F z} \right)^2 \right]} \qquad (7)$$

Eq. (7) is a special case of the expression given in Fetter & Walecka, p. 355, eq. (40.14).

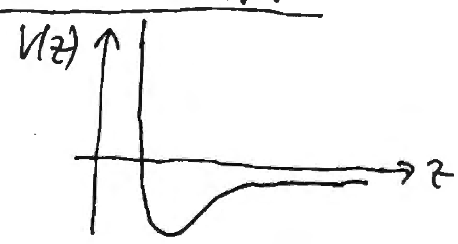
Possible applications:

1) interacting electron gas

$V(z) = \dfrac{e^2}{z}$



2) nuclear matter



energy shift in 2nd-order perturb. theory

We start from the expression given on p. 41:

$$\Delta E^{(2)} = \frac{-i}{\hbar} \langle \Phi_0 | T\{ \hat{H}_1^{(I)}(0) \cdot \int_{-\infty}^{0} dt \, \hat{H}_1^{(I)}(t) \} | \Phi_0 \rangle_{\text{connected}} =$$

$$= -\frac{i}{\hbar} \int_{-\infty}^{0} dt \, \langle \Phi_0 | T\{ \hat{H}_1^{(I)}(0) \hat{H}_1^{(I)}(t) \} | \Phi_0 \rangle_{\text{conn}}$$

Insert the interaction Hamiltonian in the I-picture at times 0 and t (notes p. 22, eq. 2); apply Wick's theorem:

$$\Delta E^{(2)} = -\frac{i}{\hbar} \int_{-\infty}^{0} dt \, \frac{1}{2} \sum_{\lambda\lambda'\mu\mu'} \int d^3x_1 \int d^3x_2 \, V(\vec{x}_1, \vec{x}_2)_{\lambda\lambda'\mu\mu'} \times$$

$$\times \frac{1}{2} \sum_{\alpha\alpha'\beta\beta'} \int d^3y_1 \int d^3y_2 \, V(\vec{y}_1, \vec{y}_2)_{\alpha\alpha'\beta\beta'} \times$$

$$\times \langle \Phi_0 | N\{ \hat{\psi}_\lambda^+(\vec{x}_1, 0) \hat{\psi}_\mu^+(\vec{x}_2, 0) \hat{\psi}_{\mu'}(\vec{x}_2, 0) \hat{\psi}_{\lambda'}(\vec{x}_1, 0) \hat{\psi}_\alpha^+(\vec{y}_1, t) \hat{\psi}_\beta^+(\vec{y}_2, t)$$

$$\times \hat{\psi}_{\beta'}(\vec{y}_2, t) \hat{\psi}_{\alpha'}(\vec{y}_1, t) \} | \Phi_0 \rangle_{\text{connected}}$$

$$+ \text{ all other connected, fully contracted terms.}$$

Again, we use

$$\underbrace{\hat{\psi}_\alpha(x) \hat{\psi}_\beta^+(y)} = - \underbrace{\hat{\psi}_\beta^+(y) \hat{\psi}_\alpha(x)} = i \, G^0_{\alpha\beta}(x, y)$$

and obtain:

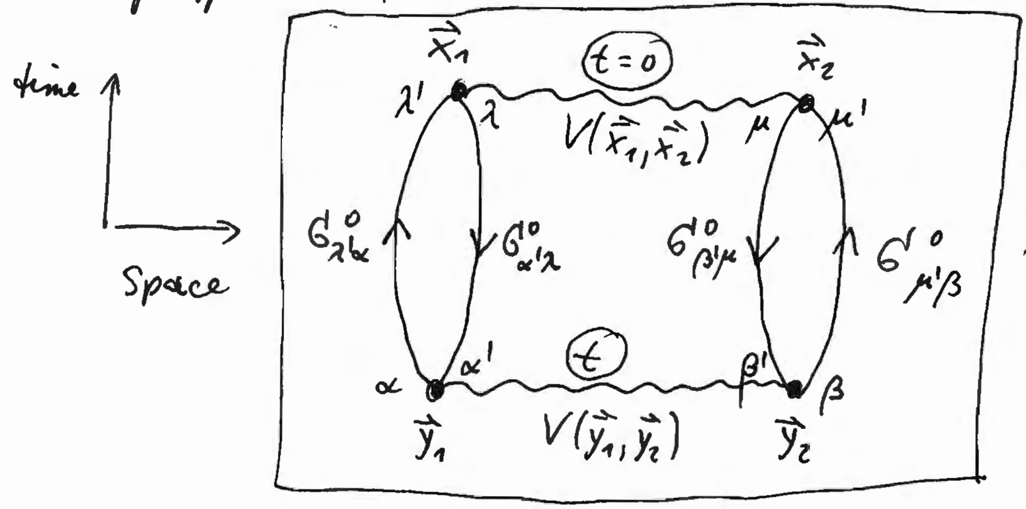
$$\Delta E^{(2)} = \left(-\frac{i}{\hbar}\right) \frac{1}{4} \int_{-\infty}^{0} dt \int d^3x_1 \int d^3x_2 \int d^3y_1 \int d^3y_2 \sum_{\lambda\lambda'\mu\mu'} \sum_{\alpha\alpha'\beta\beta'} \times$$

$$\times V(\vec{x}_1, \vec{x}_2)_{\lambda\lambda'\mu\mu'} V(\vec{y}_1, \vec{y}_2)_{\alpha\alpha'\beta\beta'} [i \, G^0_{\lambda'\alpha}(\vec{x}_1, 0; \vec{y}_1, t)] \times$$

$$\times [i \, G^0_{\mu'\beta}(\vec{x}_2, 0; \vec{y}_2, t)] [-i \, G^0_{\beta'\mu}(\vec{y}_2, t; \vec{x}_2, 0)] [-i \, G^0_{\alpha'\lambda}(\vec{y}_1, t; \vec{x}_1, 0)]$$

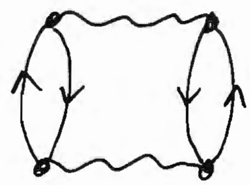
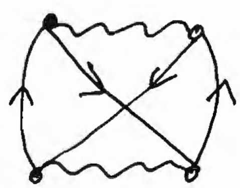
+ other terms.

In graphical form:



see FW fig. 9.22 (p.113)

According to textbook FW p.113, there is only one other connected diagram that is topologically distinct:

$\Delta E^{(2)} =$  $+$ 

"direct" "exchange"

The explicit expression for $\Delta E^{(2)}$ can be found by inserting the free propagators $i\,G^{0}_{\alpha\beta}(x,y)$ from p.14. Even for a spin-independent potential $V(\vec{x},\vec{x}')$ and for a homogeneous medium the expression is quite horrible (FW p.118):

$$\frac{\Delta E^{(2)}}{L^3} = (2\pi)^{-9} \int_0^{k_F} d^3k \int_0^{k_F} d^3p \int_{k_F}^{\infty} d^3l \int_{k_F}^{\infty} d^3n \; \delta^{(3)}(\vec{k}+\vec{p}-\vec{l}-\vec{n}) \times$$

$$\times \left[2\,\tilde{V}(\vec{l}-\vec{k})^2 - \tilde{V}(\vec{l}-\vec{k})\,\tilde{V}(\vec{p}-\vec{l}) \right] \frac{1}{\frac{\hbar^2}{2m}\left(p^2+k^2-l^2-n^2+i\eta\right)}$$

where $\tilde{V}(\vec{p})$ denotes the Fourier transform of $V(\vec{x},\vec{x}')$.

Let us briefly comment on the origin of the Fourier transforms in the last eqn. The expression we derived on p. 48 for $\Delta E^{(2)}$ contains numerous Green's functions of the form

$$G^0_{\alpha\beta}(\vec{x},t;\vec{x}',t') \propto e^{i[\vec{k}\cdot(\vec{x}-\vec{x}')-\omega_k(t-t')]}$$

and we have numerous spatial integrals involving the 2-body potentials such as

$$I = \int d^3x_1 \int d^3x_2\, V(|\vec{x}_1-\vec{x}_2|)\, e^{i(\vec{k}_1-\vec{k}_2)\cdot\vec{x}_1}\, e^{i(\vec{k}_3-\vec{k}_4)\cdot\vec{x}_2}$$

where the plane-wave factors arise from G^0. For a uniform system, we use the coord. transf. $\vec{x}_1 = \vec{r}_1+\vec{r}_2$, $\vec{x}_2 = \vec{r}_2$:

$$\left.\begin{aligned} I &= \int d^3r_1 \int d^3r_2\, V(r_1)\, e^{i(\vec{k}_1-\vec{k}_2)\cdot\vec{r}_1}\, e^{i(\vec{k}_1+\vec{k}_3-\vec{k}_2-\vec{k}_4)\cdot\vec{r}_2} \\ &= (2\pi)^3\,\delta(\vec{k}_1+\vec{k}_3-\vec{k}_2-\vec{k}_4)\,\tilde{V}(\vec{k}_1-\vec{k}_2). \end{aligned}\right\}$$

This explains where the F.T.'s of the potentials come from.

Finally, the expression for $\Delta E^{(2)}$ on p. 48 contains a time integral of the form (the ω_n-factors are from the Green's functions)

$$\left.\begin{aligned} &\int_{-\infty}^{0} dt\, e^{i\overbrace{}^{}\underbrace{(\omega_1+\omega_3-\omega_2-\omega_4)}_{=\Delta\omega}t} = \lim_{\eta\to 0}\int_{-\infty}^{0} e^{i(\Delta\omega+i\eta)t} = \\ &= \lim_{\eta\to 0}\frac{1}{i(\Delta\omega+i\eta)}. \end{aligned}\right\}$$

Expressing the frequencies $\hbar\omega_n = \frac{\hbar^2}{2m}k_n^2$ in terms of momenta, we arrive at the structure

$$\frac{1}{(\omega_1+\omega_3-\omega_2-\omega_4+i\eta)} \propto \frac{1}{(k_1^2+k_3^2-k_2^2-k_4^2+i\eta)}$$

which explains the denominator in the eq. for $\Delta E^{(2)}$ on p. 49.

Dyson's integral eqns for exact propagator $G_{\alpha\beta}$ and "proper self-energy"

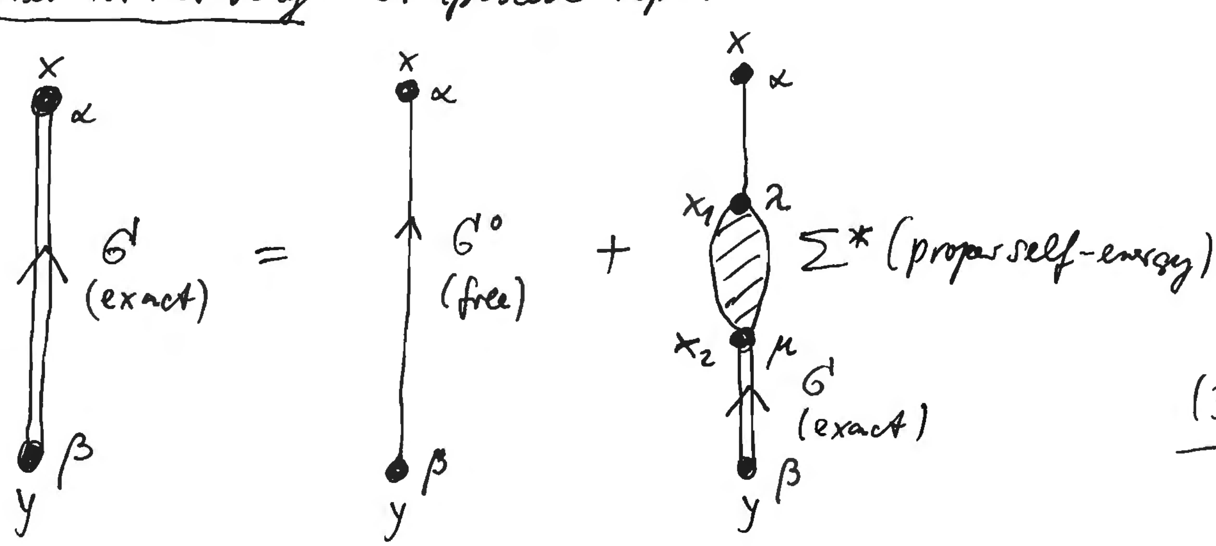
We have developed the expansion of the exact propagator $G_{\alpha\beta}$ for n^{th}-order perturbation theory

$$G_{\alpha\beta} = G^{(0)}_{\alpha\beta} + G^{(1)}_{\alpha\beta} + \dots + G^{(n)}_{\alpha\beta} + \dots \qquad \underline{(1)}$$

However, this procedure becomes impractical for higher orders ($n \gtrsim 2$) of P.T. and one must instead resort to approximation schemes. One convenient starting point for these is an integral eqn. derived by Dyson (textbook, p.106):

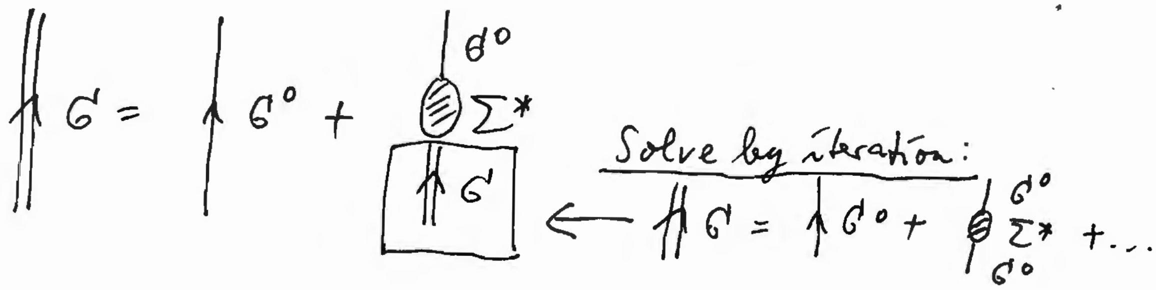
$$\left.\begin{aligned} \underbrace{G_{\alpha\beta}(x,y)}_{\text{exact propagator}} &= G^{0}_{\alpha\beta}(x,y) + \int d^4x_1 \int d^4x_2\, G^{0}_{\alpha\lambda}(x,x_1)\, \times \\ &\quad \times \underbrace{\Sigma^*}_{\text{"proper self-energy"}}(x_1,x_2)_{\lambda\mu}\, \underbrace{G_{\mu\beta}(x_2,y)}_{\text{exact propagator}} \end{aligned}\right\} \qquad \underline{(2)}$$

Note that this integral eq. contains the exact propagator on both the left-hand and right-hand side, so it must in general be solved iteratively. Graphical repr:

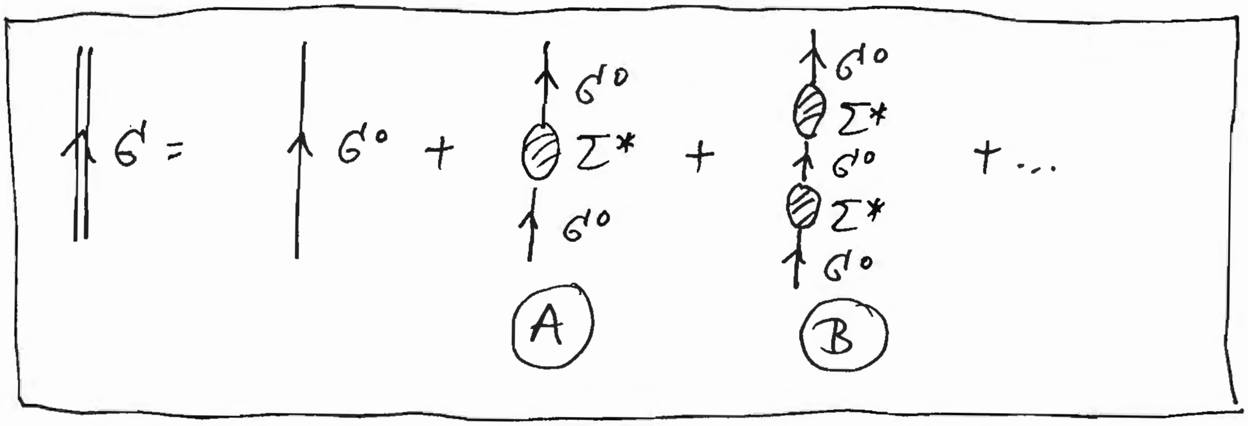


$\underline{(3)}$

Same (in more symbolic notation):

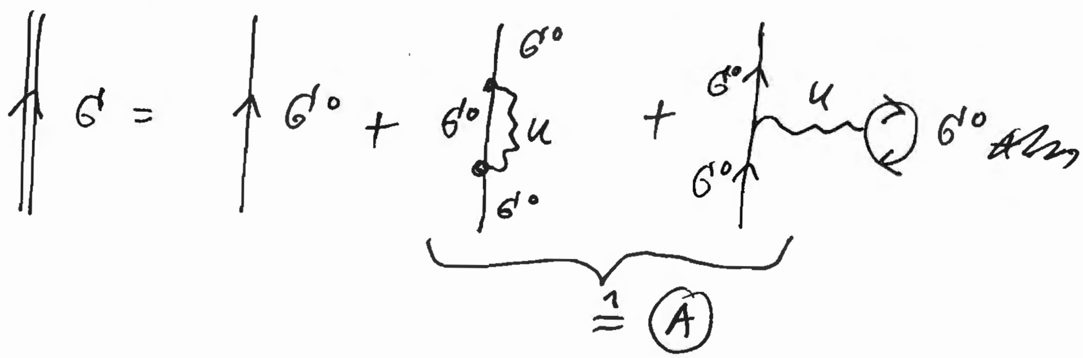


Insert iteration:



(4)

Let's compare graph (A) to the result we obtained earlier in 1st-order perturb. theory (notes p. 30):



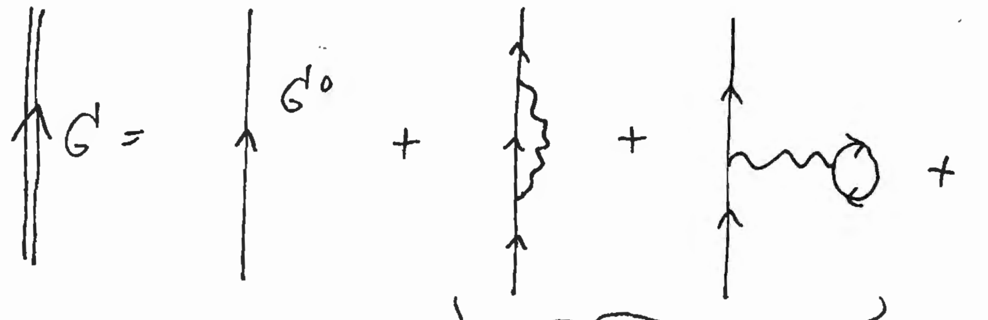
(5)

By comparison to graph (4), we can deduce the proper self-energy in 1st-order:

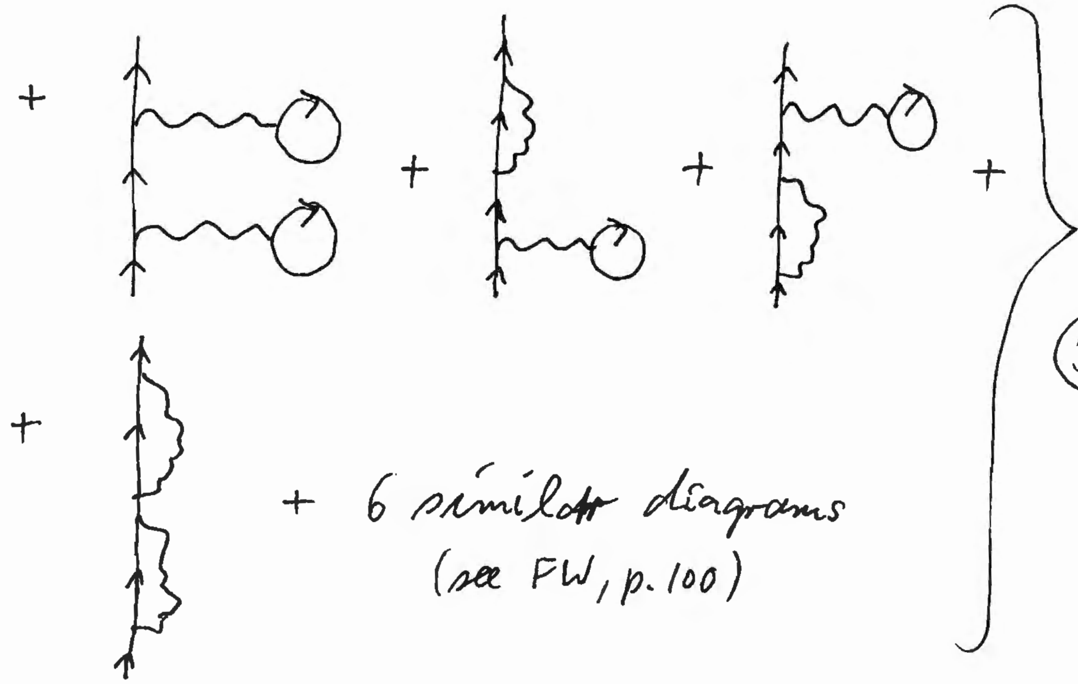
see FW, Fig. 9.15

(6)

Let us now insert (6) for $\Sigma^*_{(1)}$ into the expression (4) for G:



Ⓐ $\hat{=}\; G'^{(1)}$



+ 6 similar diagrams

(see FW, p. 100)

Ⓑ $\hat{=}\; G'^{(2)}$

The above diagrams are exactly the same as derived earlier from the perturbative expansion of G' in 1st and 2nd order

> Hence, Dyson's integral eq. (2) for the exact fermion propagator is equivalent to perturbative expansion of G'.

<u>Dyson's eq.</u> becomes particularly <u>simple</u> for <u>uniform</u> <u>systems</u>. In this case, one transforms to momentum space

$$G'_{\alpha\beta}(\vec{x},t;\vec{x}',t') \longrightarrow G'_{\alpha\beta}(\vec{x}-\vec{x}',t-t')$$

$$\xrightarrow{\text{mom. space}} \underline{\tilde{G}_{\alpha\beta}(\vec{k},\omega)}.$$

One finds, FW (9.33):

$$\boxed{G_{\alpha\beta}(\vec{k},\omega) = \delta_{\alpha\beta}\,\frac{1}{\omega - \epsilon_k^0/\hbar - \Sigma^*(\vec{k},\omega)}} \qquad \underline{(7)}$$

which expresses $G_{\alpha\beta}$ in terms of proper self-energy in mom. space via a <u>simple algebraic relation</u> (rather than integral eq!).

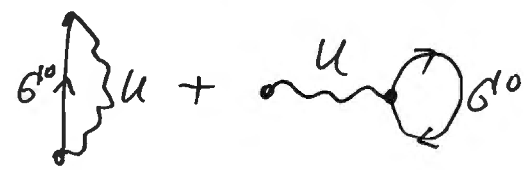
Note: propagators in mom. space is a possible term paper topic!

From Dyson's eq. to HF mean field theory

Ref: FW p. 120–126

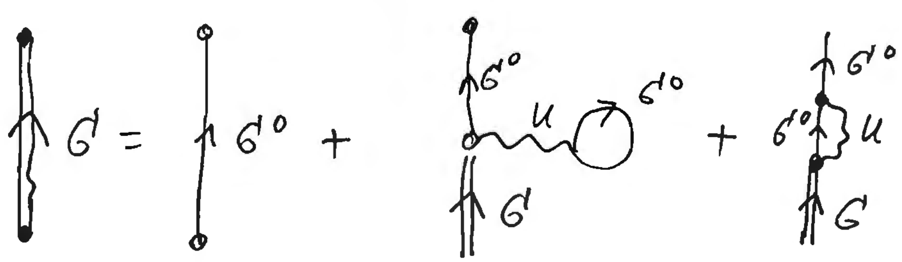
<u>First approx:</u> <u>use proper self-energy in 1st order</u>

$$\Sigma^* \approx \Sigma^*_{(1)} \overset{(6)}{=}$$

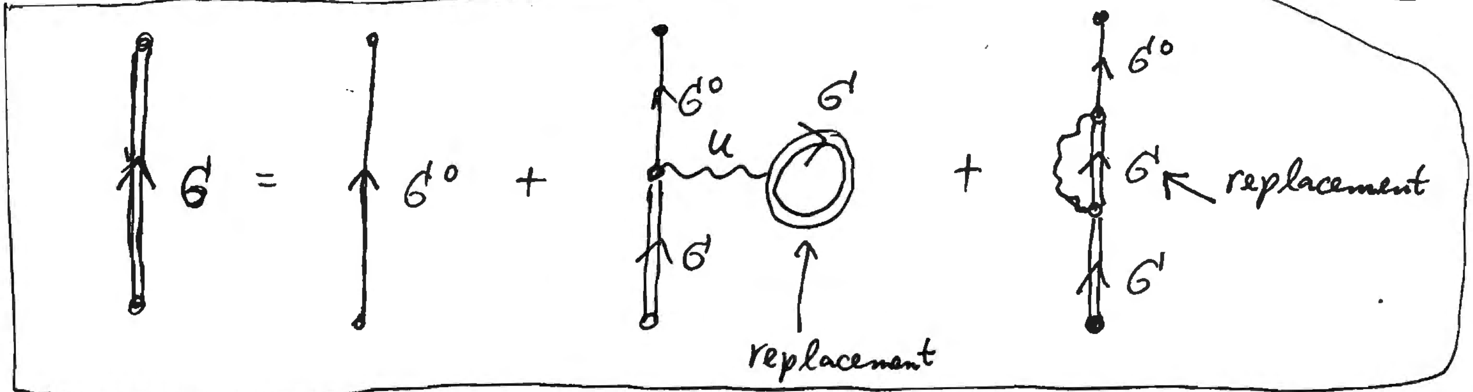


$\underline{(8)}$

Insert (8) into Dyson's eq. <u>(3)</u>:



<u>Second approximation:</u> replace <u>some</u> <u>free</u> propagators G^0 by <u>exact</u> propagator G (double fermion lines on next page).



Dyson's eq. for G in HF-approximation (FW, fig. 10.5)

This eq. for G is now self-consistent because "G both determines and is determined by the proper self-energy Σ^*" (FW p. 120).

The 2nd approx. above appears rather arbitrary, and only makes sense if one knows what one wants to obtain (HF equations).

From the Dyson eqs. above one can derive the HF-equations. Since the eqns contain the exact G, this means:

HF mean field theory corresponds to perturb. theory in ∞ order

Details of derivation, see FW p. 122–126. One finds that the proper self-energy is equal to the HF mean field potential; see FW (10.16):

$$\hbar\,\Sigma^*(\vec{x}_1,\vec{x}_1{}') = \delta(\vec{x}_1-\vec{x}_1{}')\int d^3x_2\, V(\vec{x}_1-\vec{x}_2)\,\rho(\vec{x}_2) \qquad \text{Hartree pot.}$$

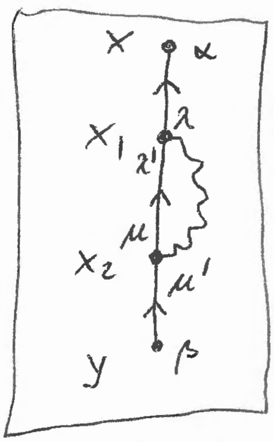
$$- V(\vec{x}_1-\vec{x}_1{}')\,\rho(\vec{x}_1,\vec{x}_1{}') . \qquad \text{Fock pot.}$$

Feynman diagrams in momentum space

Lit: FW p. 100 – 111

So far, we have derived the Feynman rules in coordinate space which allow us to write down the fermion propagator $G^{(n)}_{\alpha\beta}(x,y)$ in n-th order perturbation theory. In practice, this leads to difficulties arising from the time-ordering in the free propagators G^0.

Let us look ~~at~~ at the 1st order Feynman diagram, for example. Using the Feynman rules, one obtains (FW, eq. 9.6):



FW, Fig. 9.7

$$G^{(1)}_{\alpha\beta}(x,y) = \left(\frac{i}{\hbar}\right) \int d^4x_1 \int d^4x_2 \sum_{\lambda\lambda'\mu\mu'}$$

$$G^0_{\alpha\lambda}(x,x_1)\, U(x_1,x_2)_{\lambda\lambda',\mu\mu'}\, G^0_{\lambda'\mu}(x_1,x_2)\, G^0_{\mu'\beta}(x_2,y) \qquad (1)$$

Each one of the free Green's functions has the structure

$$i\,G^0_{\alpha\beta}(x,x') \equiv i\,G^0_{\alpha\beta}(x-x') = i\,G^0_{\alpha\beta}(\vec{x}-\vec{x}', t-t') =$$

lecture notes ch. 10, p. 14 $= \delta_{\alpha\beta}\left(\frac{1}{2\pi}\right)^3 \int d^3k\, e^{i[\vec{k}\cdot(\vec{x}-\vec{x}') - \omega_k(t-t')]}$

$$\times \left[\theta(t-t')\,\theta(k-k_F) - \theta(t'-t)\,\theta(k_F-k)\right] \qquad (2)$$

If we insert (2) into (1) we notice that the time-integrations in each one of the 3 free propagators G^0 in (1) must be split up into 2 pieces each because of the θ-functions in time. So, even in 1st order perturbation theory, this leads to many different terms!

By contrast, the Fourier transform (F.T.) with respect to time, $t \to \omega$, has a simple form which is convenient to incorporate into actual calculations.

1) For spatially inhomogeneous systems with time-indep. Hamiltonian, it is useful to consider the mixed representation

$$G_{\alpha\beta}(x,y) \equiv G_{\alpha\beta}(\vec{x},\vec{y};\underline{t_x,t_y}) \xrightarrow{F.T.} G_{\alpha\beta}(\vec{x},\vec{y};\underline{\omega})$$

i.e. F.T. only with respect to time, but ~~both~~ leave the spatial dependance.

2) We now restrict discussion to systems which are uniform and isotropic. In this case, both the exact and free Green's function depend only on the 4-d coordinate differences

$$G_{\alpha\beta}(x,y) \to \underline{G_{\alpha\beta}(x-y)} \equiv G_{\alpha\beta}(\vec{x}-\vec{y},t_x-t_y)$$ and

we can carry out a 4-d Fourier transformation. Using the notation

$$\left.\begin{aligned} k &\equiv (\vec{k},\omega) \qquad \int d^4k \equiv \int d^3k \int d\omega \\ \text{and} \qquad k\cdot x &= \vec{k}\cdot\vec{x} - \omega t \end{aligned}\right\} \quad (3)$$

we have for the exact propagator

$$\boxed{G_{\alpha\beta}(x-y) = (2\pi)^{-4}\int d^4k\, e^{ik\cdot(x-y)}\, \underline{G_{\alpha\beta}(k)}} \qquad (4)$$

which defines the 4-d F.T. $G_{\alpha\beta}(k)$.

4-d Fourier transform of free propagator, $G^0(k)$

Ref: FW, p. 72

The free propagator $iG^0(x-x')$ given in eqn. (2) contains θ-functions in time: $\theta(t-t')$ and $\theta(t'-t)$.

Using the theory of residues, one can show that the θ-function has the following integral representation:

$$\theta(T) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\omega \, \frac{e^{-i\omega T}}{\omega + i\eta} \qquad \underline{(5)}$$

Insert (5) into (2):

$$\left\{ iG^0_{\alpha\beta}(x-x') \equiv iG^0_{\alpha\beta}(\vec{x}-\vec{x}', t-t') = \delta_{\alpha\beta}\left(\frac{1}{2\pi}\right)^3\left(\frac{i}{2\pi}\right)\int d^3k\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \cdot \underbrace{e^{-i\omega_k(t-t')}}\left[\theta(k-k_F)\underbrace{\int_{-\infty}^{+\infty}\frac{e^{-i\omega(t-t')}d\omega}{\omega+i\eta}} - \theta(k_F-k)\underbrace{\int_{-\infty}^{+\infty}\frac{e^{-i\omega(t'-t)}d\omega}{\omega+i\eta}}\right]\right\}$$

Combining the underlined terms we obtain

$$\left\{\begin{aligned} iG^0_{\alpha\beta}(x-x') &= i\delta_{\alpha\beta}(2\pi)^{-4}\int d^3k\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\int_{-\infty}^{+\infty} d\omega\, \underline{f(\omega)} \\ \text{with } \underline{f(\omega)} &= \theta(k-k_F)\frac{e^{-i(\omega_k+\omega)(t-t')}}{(\omega+i\eta)} + \theta(k_F-k)\frac{e^{-i(-\omega+\omega_k)(t-t')}}{(-\omega-i\eta)}\end{aligned}\right.$$

$\underline{(6)}$

Using the variable substitutions

1st integral: $\omega' = \omega + \omega_k \quad \Rightarrow d\omega = d\omega'$

2nd integral: $\omega'' = -\omega + \omega_k \quad \Rightarrow d\omega = -d\omega''$

one finds straightforwardly

$$\int_{-\infty}^{+\infty} d\omega\, f(\omega) = \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega(t-t')}\left[\frac{\theta(k-k_F)}{\omega-\omega_k+i\eta} + \frac{\theta(k_F-k)}{\omega-\omega_k-i\eta}\right]. \qquad \underline{(7)}$$

Insert (7) into (6):

$$G^0_{\alpha\beta}(x-x') = \underset{\sim\sim}{\delta_{\alpha\beta}}(2\pi)^{-4}\int d^4k\; e^{i[\overbrace{\vec{k}\cdot(\vec{x}-\vec{x}')}^{=e^{ik\cdot(x-x')}}-\omega(t-t')]}\times$$

$$\times\underbrace{\left[\frac{\theta(k-k_F)}{\omega-\omega_k+i\eta}+\frac{\theta(k_F-k)}{\omega-\omega_k-i\eta}\right]}_{\sim\sim\sim\sim}$$

A comparison with the def. of the 4-d F.T., eq. (4), yields

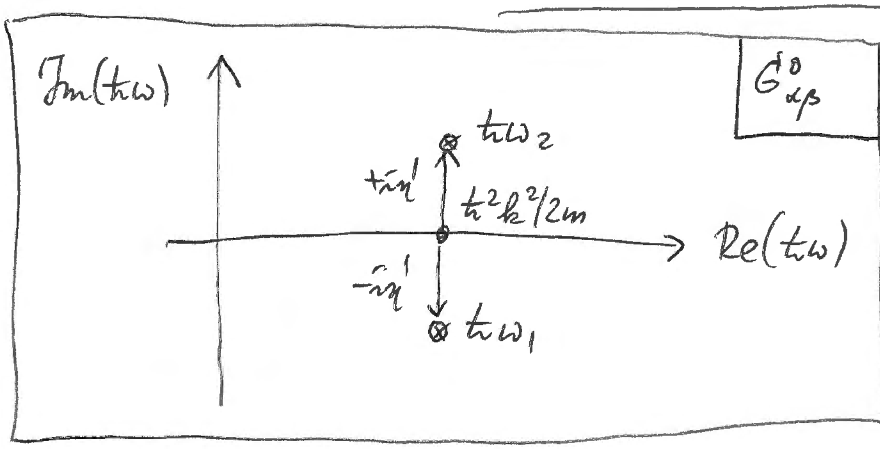
$$\boxed{G^0_{\alpha\beta}(k)\equiv G^0_{\alpha\beta}(\vec{k},\omega)=\delta_{\alpha\beta}\left[\frac{\theta(k-k_F)}{\omega-\omega_k+i\eta}+\frac{\theta(k_F-k)}{\omega-\omega_k-i\eta}\right]}$$

We observe that $G^0(k)$ has <u>poles in the complex plane</u>.

For <u>$k>k_F$</u>: pole at $\omega_1-\omega_k+i\eta=0 \rightsquigarrow \omega_1=\omega_k-i\eta$

pole is located at energy $\underline{\hbar\omega_1}=\hbar\omega_k-i\eta'=\underline{\frac{\hbar^2k^2}{2m}-i\eta'}$ ← free particle!

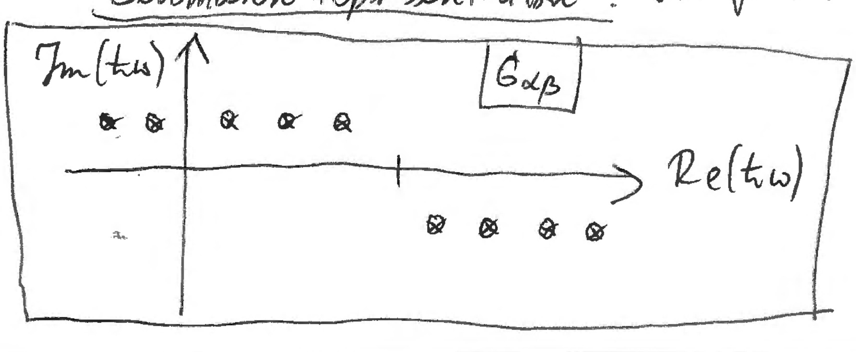
Similarly, for <u>$k<k_F$</u>: $\underline{\hbar\omega_2=\frac{\hbar^2k^2}{2m}+i\eta'}$



<u>The poles of $G^0(k)$</u> in complex ω-plane correspond to <u>s.p. energies $\hbar\omega_k=\hbar^2k^2/2m$</u> of the system!

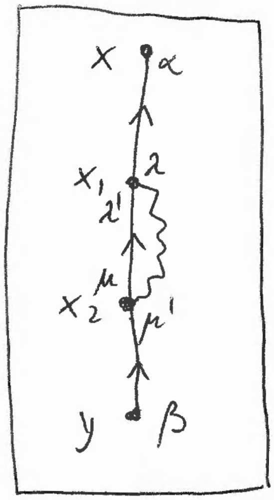
This analysis can be generalized for the <u>exact propagator $G_{\alpha\beta}$</u>

→ <u>"Lehmann representation"</u>. One finds (FW, p. 76/77)



<u>Simple poles</u> at <u>exact excitation energies</u> of the interacting system!

First-order propagator, $G^{(1)}_{\alpha\beta}(k)$, in momentum space



Our goal is to transfer the 1st-order propagator shown on the left into momentum space, $\{k\}=\{\vec{k},\omega\}$.

From p. 56:

$$\left.\begin{aligned} G^{(1)}_{\alpha\beta}(x,y) &= \left(\frac{i}{\hbar}\right)\int d^4x_1 \int d^4x_2 \sum_{\lambda\lambda'\mu\mu'} G^{0}_{\alpha\lambda}(x,x_1)\cdot \\ &\cdot U(x_1,x_2)_{\lambda\lambda'\mu\mu'}\, G^{0}_{\lambda'\mu}(x_1,x_2)\, G^{0}_{\mu'\beta}(x_2,y) \end{aligned}\right\} \underline{(9)}$$

Now insert Fourier transforms of 3 free propagators G^0, e.g.

$$G^{0}_{\alpha\lambda}(x,x_1) \equiv G^{0}_{\alpha\lambda}(x-x_1) = (2\pi)^{-4}\int d^4k\, e^{ik(x-x_1)}\, G^{0}_{\alpha\lambda}(k)$$

and for the interaction potential

$$U(x_1,x_2)_{\lambda\lambda'\mu\mu'} \equiv U(x_1-x_2)_{\dots} = (2\pi)^{-4}\int d^4q\, e^{iq(x_1-x_2)}\, U(q)_{\lambda\lambda'\mu\mu'}$$

We obtain

$$\left\{\begin{aligned} G^{(1)}_{\alpha\beta}(x,y) &= \left(\frac{i}{\hbar}\right)\underline{\int d^4x_1}\,\underset{\sim}{\int d^4x_2} \sum_{\lambda\lambda'\mu\mu'} (2\pi)^{-16}\int d^4k \int d^4p \\ &\times \int d^4p_1 \int d^4q\, G^{0}_{\alpha\lambda}(k)\, U(q)_{\lambda\lambda'\mu\mu'}\, G^{0}_{\lambda'\mu}(p)\, G^{0}_{\mu'\beta}(p_1) \\ &\times e^{i\underline{k}\cdot(x-\underline{x_1})}\, e^{i\underline{q}\cdot(x_1-\underset{\sim}{x_2})}\, e^{i\underline{p}\cdot(x_1-\underset{\sim}{x_2})}\, e^{i\underset{\sim}{p_1}\cdot(\underset{\sim}{x_2}-y)} \end{aligned}\right\} \underline{(10)}$$

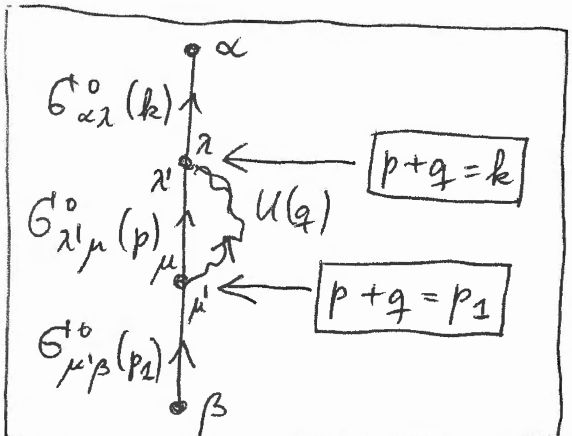
The underlined (— and ~) spatial integrations lead to 4-d delta functions:

$$\left.\begin{aligned} \text{—}:\quad & (2\pi)^{-4}\int d^4x_1\, e^{i(-k+q+p)\cdot x_1} = \delta^{(4)}(p+q-k) \\ \sim:\quad & (2\pi)^{-4}\int d^4x_2\, e^{i(-q-p+p_1)\cdot x_2} = \delta^{(4)}(p_1-q-p) \end{aligned}\right\} \underline{(11)}$$

Insert (11) into (10):

$$\left.\begin{aligned} G'^{(1)}_{\alpha\beta}(x,y) = \left(\frac{i}{\hbar}\right)(2\pi)^{-8} \int d^4k \int d^4p \int d^4p_1 \int d^4q \, \times \\ \times \sum_{\lambda\lambda'\mu\mu'} G'^{0}_{\alpha\lambda}(k)\, U(q)_{\lambda\lambda'\mu\mu'}\, G'^{0}_{\lambda'\mu}(p)\, G'^{0}_{\mu'\beta}(p_1) \\ \times e^{ik\cdot x} e^{-ip_1\cdot y}\, \underbrace{\delta^{(4)}(p+q-k)}\, \underbrace{\delta^{(4)}(p_1-q-p)} \end{aligned}\right\}$$

The $\delta^{(4)}$ functions describe 4-momentum conservation at the 2 internal vertices (x_1, x_2). We carry out the integrations over



Feynman diagram in momentum space

$$\int d^4q \;:\; \delta^{(4)} \rightarrow \underline{q = k-p}$$

$$\int d^4p_1 \;:\; \delta^{(4)} \rightarrow \underline{p_1} = p+q = p+(k-p) = \underline{k}$$

We obtain:

$$\left.\begin{aligned} G'^{(1)}_{\alpha\beta}(x,y) = \left(\frac{i}{\hbar}\right)(2\pi)^{-8} \underline{\int d^4k\, e^{ik\cdot(x-y)}} \sum_{\lambda\lambda'\mu\mu'} G'^{0}_{\alpha\lambda}(k) \,\times \\ \times \int d^4p\, U(k-p)_{\lambda\lambda'\mu\mu'}\, G'^{0}_{\lambda'\mu}(p)\, G'^{0}_{\mu'\beta}(k) \end{aligned}\right\} \quad \underline{(12)}$$

Using the def. of the F.T. of $G'^{(1)}_{\alpha\beta}$

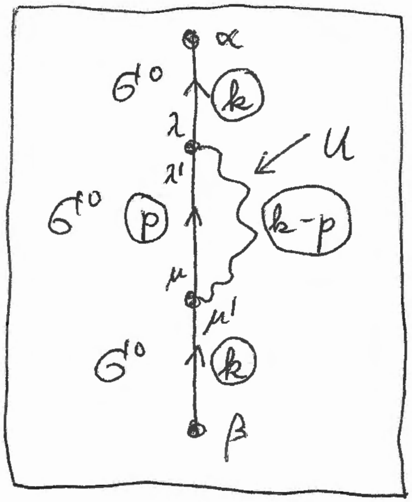
$$G'^{(1)}_{\alpha\beta}(x,y) \equiv G'^{(1)}_{\alpha\beta}(x-y) = \underline{(2\pi)^{-4} \int d^4k\, e^{ik\cdot(x-y)}}\, G'^{(1)}_{\alpha\beta}(k)$$

we find by comparison with (12):

$$\boxed{\begin{aligned} G'^{(1)}_{\alpha\beta}(k) = \left(\frac{i}{\hbar}\right) \sum_{\lambda\lambda'\mu\mu'} G'^{0}_{\alpha\lambda}(k) \left[(2\pi)^{-4} \int d^4p\, U(k-p)_{\lambda\lambda'\mu\mu'} \right. \\ \left. \times\, G'^{0}_{\lambda'\mu}(p)\right] G'^{0}_{\mu'\beta}(k) \end{aligned}}$$

$G^{(1)}$ in momentum space

We draw 4-d momentum space Feynman diagrams one more time, emphasizing the 4-momenta of the free propagators G^0 and 4-pot. U:



This diagram describes scattering of one particle (momentum $k = \{\vec{k}, \omega\}$) in the medium from other particles, with finite momentum transfer $(k-p)$.