Section 2.2a | Basic theoretical concepts

Structure of nuclear many-body Hamiltonian

We consider here atomic nuclei in their ground state or at excitation energies up to about $E^* \simeq 20\,MeV$. Under these conditions, the nucleons "see" each other as point particles which may be characterized by their position vectors $\vec{r}_i$, their spin projections $s_z^{(i)} = \pm\frac{1}{2}$ and their isospin projections $t_z^{(i)} = \pm\frac{1}{2}$. The isospin formalism will be explained in one of the following sections, we note here only that

$$\text{proton} = |p\rangle \equiv \text{"nucleon" with } t_z = +\tfrac{1}{2} \left(\text{"isospin up"}\right)$$

$$\text{neutron} = |n\rangle \equiv \text{"nucleon" with } t_z = -\tfrac{1}{2} \left(\text{"isospin down"}\right)$$

Degrees of freedom for nucleons:

$$\boxed{x_i = \left(\vec{r}_i, s_z^{(i)}, t_z^{(i)}\right); \qquad i = 1, \dots, A}$$

The many-body nuclear Hamiltonian can be written in the form

$$H(x_1, \dots, x_A) = \underbrace{\sum_{i=1}^{A} -\frac{\hbar^2}{2m} \nabla_i^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2} \sum_{i,j=1}^{Z} v_{ij}^{(2)\,coul}(x_i, x_j)}_{\text{Coulomb energy (p's only)}}$$

$$+ \frac{1}{2} \sum_{i,j=1}^{A} v_{ij}^{(2)\,nucl}(x_i, x_j) + \frac{1}{6} \sum_{i,j,k=1}^{A} v_{ijk}^{(3)\,nucl}(x_i, x_j, x_k)$$

We see several types of operators in H:

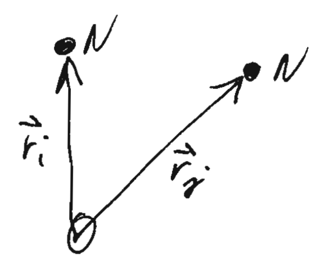
$$\sum_{i=1}^{N} A(x_i) \;\hat{=}\; \boxed{\text{one-body operator}} \quad (\text{operates on } x_i \text{ only})$$

Example: kinetic energy operator.

$$\sum_{j,i=1}^{N} B(x_i, x_j) \;\hat{=}\; \boxed{\text{two-body operator}} \quad (\text{operates on pair } x_i, x_j)$$

Examples: Coulomb int. between 2 protons

$$v_{ij}^{(2)\,\text{Coul}}(x_i, x_j) = \frac{e^2}{|\vec{r}_i - \vec{r}_j|}$$

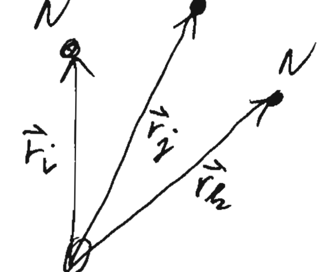


There is also a 2-body interaction which arises from the strong nuclear force; we will see that its mathematical structure is much more complicated, in particular due to spin/isospin-dependance

$$v_{ij}^{(2)\,\text{nucl}}(x_i, x_j) = v_{ij}^{(2)\,\text{nucl}}\left(\vec{r}_i, s_z^{(i)}, t_z^{(i)}; \vec{r}_j, s_z^{(j)}, t_z^{(j)}\right)$$

Details, see Section 3.1; in addition, in nuclear theory we must even consider three-body interactions of the type

$$\sum_{k,j,i=1}^{N} C(x_i, x_j, x_k) \;\hat{=}\; \boxed{\text{three-body operator}}$$



These 3-body interactions contribute about 10% to nuclear binding energy; they arise from <u>multiple</u> meson exchange.

Assume now that we know $v_{ij}^{(2)}$ and $v_{ijk}^{(3)}$; then the many-body Hamiltonian $H(x_1, \dots, x_A)$ is completely determined and we may write down the <u>stationary many-particle Schrödinger equation</u>:

$$\boxed{\begin{aligned} H(x_1, x_2, \dots, x_A)\, \Psi_n(x_1, x_2, \dots, x_A) = \\ = E_n\, \Psi_n(x_1, x_2, \dots, x_A) \end{aligned}}$$

<u>Mathematical complexity of quantum many-body problem</u>

Consider a heavy nucleus with $A \approx 200$ particles, e.g. $^{208}_{82}Pb$. In this case, H contains ≈ 200 Laplacians ∇_i^2 $(i = 1, \dots, A)$; furthermore $v_{ij}^{(2)\,mod}$ is momentum-dependent and contains additional gradient operators $\vec{p}_i = -i\hbar\nabla_i$ etc. Finally, the many-body WF consists of 5 degrees of freedom per nucleon

$$\Psi_n = \Psi_n\left(\vec{r}_1, s_z^{(1)}, t_z^{(1)}; \dots, \vec{r}_A, s_z^{(A)}, t_z^{(A)}\right)$$

i.e. about 1000 degrees of freedom for $A \simeq 200$!!

Imagine we try to solve the many-body Schrödinger eq. on a powerful supercomputer, without any approximations. Even the storage of one single many-body WF would be impossible: suppose, we discretize space with 30 grid points in x-direction, 30 points in y- and z-direction; the <u>storage of this one WF</u> would require an <u>array of size</u>

$$\approx (\underset{x_i}{30} \times \underset{y_i}{30} \times \underset{z_i}{30} \times \underset{s_z^{(i)}}{2} \times \underset{t_z^{(i)}}{2})^{200} = (1.08 \times 10^5)^{200} \approx 10^{1000}$$

Good luck!

Fortunately, the problem can be reduced somewhat in size, because the nucleons are identical, but the basic conclusion is the same. Only for the lightest nuclei (up to $A \leq 16$) can one solve the many-body diff. eqns by "brute force" on a supercomputer which takes ~ months [examples in Section 3.2].

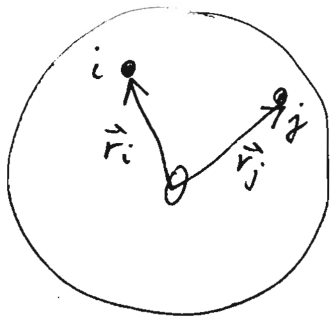
Identical particles : fermions and bosons

Lit: Gordon Baym, QM, ch. 18

As far as the dominant strong interaction is concerned, the "nucleons" (p, n) are simply two different states, $t_z = \pm\frac{1}{2}$, of the same particle and the N-N and N-N-N interactions are invariant under rotations in isospin-space (details Section 3.1). The Coulomb-int. breaks this isospin symmetry, and this effect is usually neglected in the many-body QM formalism. With these caveats, we consider Hamiltonians of the form

$$H(x_1, x_2, \dots, x_A) \qquad x_i = (\vec{r}_i, s_z^{(i)}, t_z^{(i)}).$$

The <u>nucleons</u> are <u>identical</u>. In classical physics, the identity of particles leads only to fairly trivial modifications (same mass etc.), but in QM it has profound consequences, as we will now show.



For identical particles, the many-body Hamiltonian must be invariant under the exchange of any two particle coordinates x_i and x_j, i.e.

$$\boxed{H(\dots, x_i, \dots, x_j, \dots) = H(\dots, x_j, \dots, x_i, \dots)} \quad (1)$$

We define the exchange operator P_{ij} for the particle coordinates

$$\boxed{P_{ij} =: \quad x_i \Leftrightarrow x_j} \quad (2)$$

and consider the following expression

$$\left.\begin{aligned} &P_{ij}\, H(\dots, x_i, \dots, x_j \dots)\, \psi(\dots, x_i, \dots, x_j, \dots) \overset{def}{=} \\ &H(\dots, x_j, \dots, x_i, \dots)\, \psi(\dots, x_j, \dots, x_i) = \\ &\qquad\quad \| \text{ eq (1)} \qquad\qquad\qquad \| \text{ eq.(2)} \\ &= H(\dots, x_i, \dots, x_j, \dots)\; P_{ij}\, \psi(\dots, x_i, \dots, x_j) \end{aligned}\right\} \quad (3)$$

From eq.(3), we conclude

$$P_{ij}\, H\psi - H P_{ij}\, \psi = 0 \quad \Rightarrow \quad \boxed{[P_{ij}, H] = 0} \quad (4)$$

Eq.(4) states that it is possible to construct a simultaneous set of eigenfunctions of the operators P_{ij} and H, i.e.

$$\left.\begin{aligned} P_{ij}\,\psi_{p,E} &= p\,\psi_{p,E} \\ H\,\psi_{p,E} &= E\,\psi_{p,E} \end{aligned}\right\} \quad \underline{(5)}$$

<u>Theorem:</u> The eigenvalues of P_{ij} are $p = \pm 1$.

Proof: $\psi = \underbrace{P_{ij}^2}_{1}\psi = p^2\psi \Rightarrow p^2 = 1 \Rightarrow p = \pm 1.$

<u>a) Totally symmetric WFs : bosons ($p=+1$)</u>

For the case $p=+1$, we have from (5)

$$P_{ij}\,\psi_{p=+1,E}(\ldots, x_i, \ldots, x_j, \ldots) \stackrel{def}{=} \underline{\psi_{p=+1,E}(\ldots, x_j, \ldots, x_i, \ldots)}$$

$$\underline{= (+1)\cdot\psi_{p=+1,E}(\ldots, x_i, \ldots, x_j, \ldots)} \qquad \underline{(6)}$$

i.e. the <u>many-body WF is totally symmetric</u> under $x_i \leftrightarrow x_j$ interchange. <u>Particles</u> with this property are called <u>bosons.</u>

<u>b) Totally antisymmetric WFs : fermions ($p=-1$)</u>

For the case $p=-1$, we have from (5)

$$P_{ij}\,\psi_{p=-1,E}(\ldots, x_i, \ldots, x_j, \ldots) \stackrel{def}{=} \underline{\psi_{p=-1,E}(\ldots, x_j, \ldots, x_i, \ldots)}$$

$$\underline{= (-1)\cdot\psi_{p=-1,E}(\ldots, x_i, \ldots, x_j, \ldots)} \qquad \underline{(7)}$$

i.e. the <u>many-body WF is totally antisymmetric</u> under $x_i \leftrightarrow x_j$ interchange. Such <u>particles</u> are called <u>fermions.</u>

Note: Eq. (5) shows that there exist solutions with $p = \pm 1$, but the most general solution could be of the form

$$\psi = a\,\psi_{p=+1} + b\,\psi_{p=-1}.$$

However, it is an exp. fact that many-particle systems are always in an eigenstate of P_{ij} with $p=+1$ or $p=-1$, so the most general WF is either totally symmetric or antisymmetric.

$\boxed{\text{Spin-statistic theorem (W. Pauli, 1940)}}$

Particles with spins $s = 0, 1, 2, \ldots \Rightarrow p = +1$ & WF sym. (bosons)

Particles with spins $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \Rightarrow p = -1$ & WF antisym. (fermions)

A formal proof requires quantum field theory (see e.g. Itzykson & Zuber, QFT, p. 149)

$\Longrightarrow$ Nucleons are fermions with totally antisym. many-body WFs.

One-body Hamiltonians (e.g. nuclear shell model) and Slater determinants

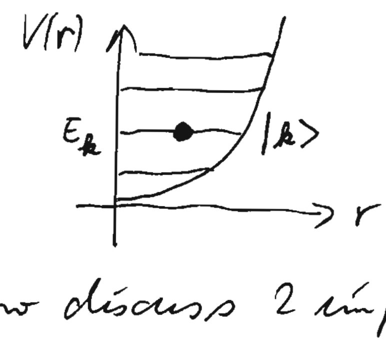
Let us study the following model problem

$$H^{(1)} = \sum_{i=1}^{A} \left(-\frac{\hbar^2}{2m} \nabla_i^2 + \frac{1}{2} m \omega^2 r_i^2 \right) = \sum_{i=1}^{A} h^{(1)}(i)$$

This problem is much simpler than the original H, because the two-body and three-body interactions $v_{ij}^{(2)}$ and $v_{ijk}^{(3)}$ have been replaced by a

<u>"one-body potential" $v^{(1)}(i)$</u>

which is often called the "<u>mean field</u>". We stress that such a mean field does not really exist, but it is possible to approximate a given many-body H in this form. In fact, in the socalled <u>Hartree-Fock theory</u> one <u>derives</u> $v_i^{(1)}$ from the given $v_{ij}^{(2)}$ and $v_{ijk}^{(3)}$!
The phenomenological shell model potential given above <u>assumes</u> that this mean field potential is a 3-D harmonic oscillator + spin-orbit term.



We now discuss 2 important theorems:

Theorem 1: Consider a many-body quantum system of N particles described by a one-body Hamiltonian of the form

$$H^{(1)} = \sum_{i=1}^{N} h^{(1)}(x_i) \qquad (*)$$

Suppose, the single-particle Hamiltonian problem can be solved, i.e.

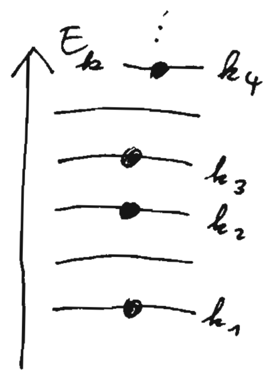
$$h^{(1)}(x)\,\varphi_k(x) = E_k\,\varphi_k(x) \qquad (**)$$

Then, if the N particles are <u>distinguishable</u>, a <u>particular</u> solution of

$$\boxed{H^{(1)}\psi(x_1,\dots,x_N) = E\,\psi(x_1,\dots,x_N)}$$

is given by the <u>s.p. WF product</u>

$$\boxed{\psi(x_1,\dots,x_N) = \varphi_{k_1}(x_1)\,\varphi_{k_2}(x_2)\cdots\varphi_{k_N}(x_N)}$$



where $(k_1, k_2, \dots, k_N)$ are the occupied s.p. states and the <u>total energy of the system</u> is

$$\boxed{E = \sum_{i=1}^{N} E_{k_i}}$$

We stress that these statements are only true for simplified <u>one-body</u> Hamiltonians.

<u>Proof:</u> $H^{(1)}\psi(x_1,\dots,x_N) = \sum_{i=1}^{N} h^{(1)}(x_i)\left[\varphi_{k_1}(x_1)\cdots\varphi_{k_i}(x_i)\cdots\right.$

$$\left.\cdots\varphi_{k_N}(x_N)\right] = \sum_{i=1}^{N}\left[\varphi_{k_1}\cdots\varphi_{k_{i-1}}\varphi_{k_{i+1}}\cdots\varphi_{k_N}\right]\underbrace{h^{(1)}(x_i)\varphi_{k_i}(x_i)}_{\overset{(**)}{=}\,E_{k_i}\varphi_{k_i}(x_i)}$$

$$= \sum_{i=1}^{N} E_{k_i} \underbrace{[\varphi_{k_1} \cdots \varphi_{k_N}]}_{= \psi(x_1, \ldots, x_N)} = \underbrace{\left(\sum_{i=1}^{N} E_{k_i} \right)}_{= E} \psi(x_1, \ldots, x_N)$$

q.e.d.

We generalize this now to the important case of indistinguishable particles with $p = -1$ (fermions):

$\boxed{\text{Theorem 2:}}$ Same situation as in Theorem 1, except that we have N identical fermions.

In this case, a particular solution of the many-body problem is given by a totally anti-symmetrized product of s.p. WF's, i.e.

$$\psi(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \sum_{\nu} (-1)^{\nu} P_{\nu}\, \varphi_{k_1}(x_1)\, \varphi_{k_2}(x_2) \cdots \varphi_{k_N}(x_N)$$

where the permutation operator P_ν carries out an arbitrary permutation of the indices $(1, 2, \ldots, N)$. The sign $(-1)^\nu$ is $(+1)$ for even permutations and (-1) for odd permutations. From the theory of determinants one infers that the above many-body WF can be written in form of a determinant called "Slater determinant"

$$\psi(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \varphi_{k_1}(x_1) & \varphi_{k_1}(x_2) & \cdots & \varphi_{k_1}(x_N) \\ \varphi_{k_2}(x_1) & \varphi_{k_2}(x_2) & \cdots & \varphi_{k_2}(x_N) \\ \vdots & & & \\ \varphi_{k_N}(x_1) & \varphi_{k_N}(x_2) & \cdots & \varphi_{k_N}(x_N) \end{vmatrix}$$

or, in shorthand form,

$$\psi(1,\dots,N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \varphi_1(1) & \varphi_1(2) & \cdots & \varphi_1(N) \\ \varphi_2(1) & \varphi_2(2) & \cdots & \varphi_2(N) \\ \vdots & & & \\ \varphi_N(1) & \varphi_N(2) & \cdots & \varphi_N(N) \end{vmatrix}$$

Note that the Slater determinant incorporates the Pauli principle: if any two quantum states are the same, e.g. $|\varphi_{k_i}\rangle \equiv |\varphi_{k_j}\rangle$, the determinant vanishes, which means that this many-body state does not exist!

Sketch of proof: $H^{(1)} \psi(x_1,\dots,x_i,\dots,x_N) =$

$$= \sum_{i=1}^{N} h^{(1)}(x_i) \frac{1}{\sqrt{N!}} \begin{vmatrix} \cdots & \varphi_1(i) & \cdots \\ \cdots & \varphi_2(i) & \cdots \\ & \vdots & \\ \cdots & \varphi_N(i) & \cdots \end{vmatrix} = \dots$$

The Hamiltonian acts only on one column of the determinant at a time. Use determinantal properties ....

Important conclusion for nuclear shell model

The many-body WF's are Slater determinants consisting of s.p. WF's of the 3-D harmonic oscillator:

$$\left. \begin{aligned} & h(\vec{r}) = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 r^2 \\ & h(\vec{r})\varphi_k(\vec{r}) = E_k \varphi_k(\vec{r}) \quad \text{with} \\ & \varphi_k(\vec{r}) \equiv \varphi_{nlm}(r,\theta,\varphi) = R_{nl}(r) Y_{lm}(\theta,\varphi) \end{aligned} \right\}.$$

Spin and isospin formalism

Lit: Greiner & Maruhn, Nuclear Models, p. 39–41

Since protons and neutrons have almost the same mass, one might regard them as two independent states of the "nucleon". In analogy to spin-space, one introduces a 2-D isospin-space with the basis vectors

proton $|p\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ "isospin up"

neutron $|n\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ "isospin down"

A general nucleon state vector in this space has the form

$$|\psi\rangle = \begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix} = \psi_p\begin{pmatrix}1\\0\end{pmatrix} + \psi_n\begin{pmatrix}0\\1\end{pmatrix} = \psi_p|p\rangle + \psi_n|n\rangle.$$

prob. amplitudes

One now proceeds in complete analogy to spin-$\frac{1}{2}$ formalism (see e.g. Shankar, QM):

$$\boxed{\vec{S} = \frac{\hbar}{2}\vec{\sigma} \quad \text{Spin}} \longrightarrow \boxed{\vec{t} = \frac{1}{2}\vec{\tau} \quad \text{isospin}}$$

Note: no factor $\hbar$

with $\boxed{\tau_x \equiv \sigma_x, \tau_y \equiv \sigma_y, \tau_z \equiv \sigma_z \text{ Pauli matrices}}$

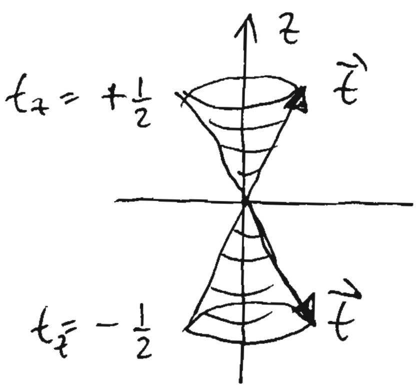
Analogies

spin	isospin
$\vec{S}^2 = \frac{\hbar^2}{4}\vec{\sigma}^2 =$	$\vec{t}^2 = \frac{1}{4}\vec{\tau}^2 =$
$= \frac{\hbar^2}{4}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2)$	$= \frac{1}{4}(\tau_x^2 + \tau_y^2 + \tau_z^2) =$
$= \hbar^2 \frac{3}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix} = \hbar^2 s(s+1)\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$= \frac{3}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix} = t(t+1)\begin{pmatrix}1&0\\0&1\end{pmatrix}$
with $s = \frac{1}{2}$	with $t = \frac{1}{2}$

Analogies (cont.)

Spin	isospin
$[\vec{s}^{\,2}, s_z] = 0 \Rightarrow \exists$ eigenstates $\|s, s_z\rangle$ with $\left.\begin{array}{l} \vec{s}^{\,2}\|s,s_z\rangle = \hbar^2 s(s+1)\|s,s_z\rangle \\ s_z\|s,s_z\rangle = \hbar s_z \|s,s_z\rangle \end{array}\right\}$ with $\underline{s=\frac{1}{2},\ s_z = \pm\frac{1}{2}}$	$[\vec{t}^{\,2}, t_z] = 0 \Rightarrow$ eigenstates $\|t, t_z\rangle$ with $\left.\begin{array}{l} \vec{t}^{\,2}\|t,t_z\rangle = t(t+1)\|t,t_z\rangle \\ t_z\|t,t_z\rangle = t_z \|t,t_z\rangle \end{array}\right\}$ with $\underline{t=\frac{1}{2},\ t_z = \pm\frac{1}{2}}$

"Geometric" interpretation for isospin



length of $\vec{t}$: $|\vec{t}| = \sqrt{t(t+1)} = \sqrt{\frac{3}{4}}$

states: $|p\rangle = |t=\frac{1}{2}, t_z = +\frac{1}{2}\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$|n\rangle = |t=\frac{1}{2}, t_z = -\frac{1}{2}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

The isospin operator t_z is useful for representing the electric charges of the proton ($q=+e$) and the neutron ($q=0$). Obviously it is true for both types of nucleons that

$$\boxed{\underbrace{q}_{\text{nucleon charge}} = \underbrace{+e}_{\text{proton charge}}\,(\underbrace{t_z}_{\text{isospin operator}} + \tfrac{1}{2})}$$

This allows us to express the Coulomb interaction of the protons, see notes p. 1 and 2, in terms of a sum over <u>all</u> nucleons ($i = 1, \dots, A$):

$$V_c = \frac{1}{2} \sum_{i,j=1}^{Z} \frac{e^2}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2} \sum_{i,j=1}^{A} \frac{e^2 (t_z^{(i)} + \frac{1}{2})(t_z^{(j)} + \frac{1}{2})}{|\vec{r}_i - \vec{r}_j|}$$

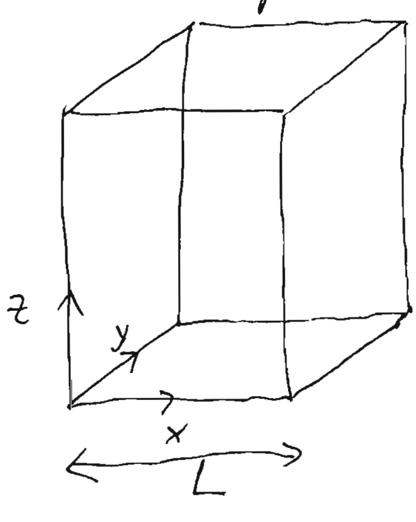
Practical application of isospin formalism

In Section 3.1, ~~the~~ we will apply this formalism to nucleon-nucleon scattering. The N-N interaction potential $v_{NN}^{(2)}(x_i, x_j)$ will be isospin-dependent, as will be the state vectors of the 2-nucleon system.

Simple illustration of spin-isospin formalism: free Fermi gas model of nucleus

Ref: Bohr & Mottelson, Nuclear structure, Vol. 1, p. 139-141

The free Fermi gas model allows us to estimate the Fermi momentum $p_F = \hbar k_F$ and the Fermi energy $E_F = p_F^2 / 2m$ as a function of nuclear matter density ρ_0. It also provides a simple explanation of the symmetry energy in the semi-empirical mass formula.



The nucleons move freely in a (large) cube of length L, with volume $V = L^3$.

One-body Hamiltonian:

$$H = \sum_{i=1}^{A} h(x_i) \quad \text{with}$$

the s.p. Hamiltonians

$$h(\vec{r}) = -\frac{\hbar^2}{2m}\nabla^2$$

and eigenstates

$$\varphi(\vec{r}, s_z, t_z) = \underbrace{L^{-3/2}\, e^{+i\vec{k}\cdot\vec{r}}}_{\text{plane wave}}\; \underset{\substack{\downarrow \\ \text{spin}}}{\chi_{s_z}}\; \underset{\substack{\downarrow \\ \text{isospin}}}{\zeta_{t_z}}\,.$$

Require <u>periodic boundary conditions</u> <u>(period L)</u>; this leads to a <u>discrete</u> momentum and energy <u>spectrum</u>:

$$k_x = \frac{2\pi}{L} n_x, \quad k_y = \frac{2\pi}{L} n_y, \quad k_z = \frac{2\pi}{L} n_z \quad \text{with}$$

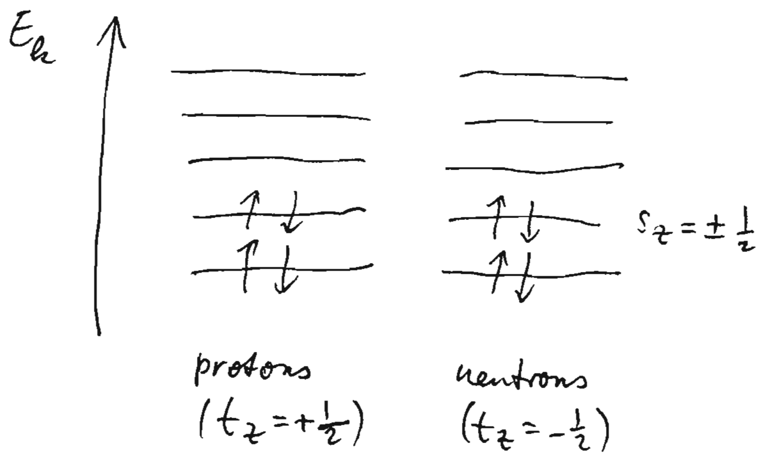
$$\left.\begin{matrix} n_x \\ n_y \\ n_z \end{matrix}\right\} = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow \boxed{\text{momenta: } \vec{p} = \hbar\vec{k} = \hbar\,\frac{2\pi}{L}(n_x, n_y, n_z)}$$

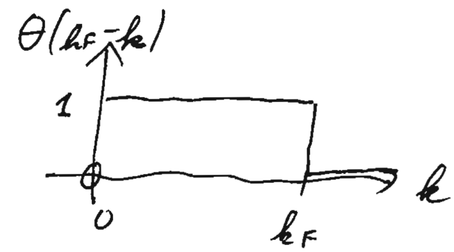
and associated s.p. energies

$$\boxed{E_{|\vec{k}|} = E_k = \frac{\hbar^2}{2m}\left(k_x^2 + k_y^2 + k_z^2\right) = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2\left(n_x^2 + n_y^2 + n_z^2\right)}$$

<u>Note</u>: E_k are quantized and independent of spin or isospin, so we have <u>4-fold degeneracy</u>:



We now derive the important relation between Fermi momentum and density. For a given isospin t_z, call the particle ~~#~~ number N_{t_z}:



$$N_{t_z} = \sum_{s_z=\pm\frac{1}{2}} \sum_{\vec{k}} \underbrace{\theta}_{\text{"theta function"}}(k_F - \underbrace{|\vec{k}|}_{=k})$$

$$\Rightarrow N_{t_z} = 2\cdot \underbrace{\sum_{\vec{k}} \theta(k_F - |\vec{k}|)}_{=f(\vec{k})} \qquad \underline{(1)}$$

In the limit of large L, we can convert this to an integral

$$\sum_{\vec{k}} f(\vec{k}) = \sum_{n_x, n_y, n_z} f\left(\frac{2\pi}{L}\vec{n}\right) \xrightarrow[L\to\infty]{} \int dn_x \int dn_y \int dn_z\, f\left(\frac{2\pi\vec{n}}{L}\right)$$

Using $dn_x = \frac{L}{2\pi} dk_x$ etc. this becomes

$$\boxed{\sum_{k} f(\vec{k}) = \left(\frac{L}{2\pi}\right)^3 \int dk_x \int dk_y \int dk_z\, f(\vec{k}) = \left(\frac{L}{2\pi}\right)^3 \int d^3k\, f(\vec{k})} \quad \underline{(2)}$$

Inserting (2) in (1) we obtain

$$\underline{N_{t_z}} = 2\cdot\left(\frac{L}{2\pi}\right)^3 \int d^3k\, \theta(k_F - |\vec{k}|) = 2\cdot\frac{L^3}{8\pi^3} \underbrace{\int d\Omega_k}_{=4\pi} \int_0^{\infty} k^2 dk\, \theta(k_F - k)$$

$$= \frac{L^3}{\pi^2}\int_0^{k_F} k^2 dk = \underline{\frac{L^3}{\pi^2}\frac{k_F^3}{3}}$$

Solving for k_F, we obtain $\boxed{k_F^{(t_z)} = \left(3\pi^2\cdot\left(\frac{N_{t_z}}{L^3}\right)\right)^{1/3}}$

$\Rightarrow$ For protons ($t_z = +\frac{1}{2}$) and $N_{+1/2} = Z$:

$$\underline{k_F^{(p)}} = \left(3\pi^2\frac{Z}{L^3}\right)^{1/3} = \underline{\left(3\pi^2\cdot\rho_0^{(p)}\right)^{1/3}}$$

Introducing the matter density (see Section 2.1, page 4)

$$\rho_0 = \frac{A}{L^3} \approx 0.17\ \mathrm{fm}^{-3}$$

we can write $\rho_0^{(p)} = \frac{Z}{A}\rho_0$ and $\rho_0^{(n)} = \frac{N}{A}\rho_0$.

For neutrons with $t_z = -\frac{1}{2}$ and $N_{-1/2} = N$ we obtain

$$\underline{k_F^{(n)} = \left(3\pi^2 \rho_0^{(n)}\right)^{1/3}.}$$

$\Rightarrow$ For symmetric nuclear matter with $N = Z = \frac{A}{2}$ we obtain this average value for protons and neutrons:

$$k_F \approx \left(3\pi^2 \cdot \frac{\rho_0}{2}\right)^{1/3}$$

Inserting $\rho_0 = 0.17\ \mathrm{fm}^{-3}$ we obtain the important estimate for the <u>Fermi momentum in nuclear matter</u>:

$$\boxed{k_F \approx 1.36\ \mathrm{fm}^{-1} \quad \text{and} \quad p_F = \hbar k_F}$$

In a similar way, one can derive the corresponding Fermi energy (for given isospin t_z) and the total energy of the system:

$$E_{tot}^{(t_z)} = \sum_{s_z = \pm\frac{1}{2}} \sum_{\vec{k}} \frac{\hbar^2 \vec{k}^2}{2m} = 2 \cdot \underbrace{\sum_{\vec{k}} \frac{\hbar^2 \vec{k}^2}{2m}}_{= f(\vec{k})}$$

Using eq. (2) on p. 16 one finds ($\rightarrow$ homework!)

$$E_{tot}^{(t_z)} = \frac{L^3}{\pi^2} \frac{\hbar^2}{2m} \frac{[k_F^{(t_z)}]^5}{5} = \frac{3}{5} N_{t_z} \frac{\hbar^2 [k_F^{(t_z)}]^2}{2m}$$

Introducing the <u>Fermi energy</u>

$$\boxed{\varepsilon_F^{(t_z)} = \frac{\hbar^2 [k_F^{(t_z)}]^2}{2m}}$$

we obtain the simple result for the <u>total Fermi gas energy:</u>

$$\boxed{E_{tot}^{(t_z)} = \frac{3}{5} N_{t_z} \cdot \varepsilon_F^{(t_z)}}$$

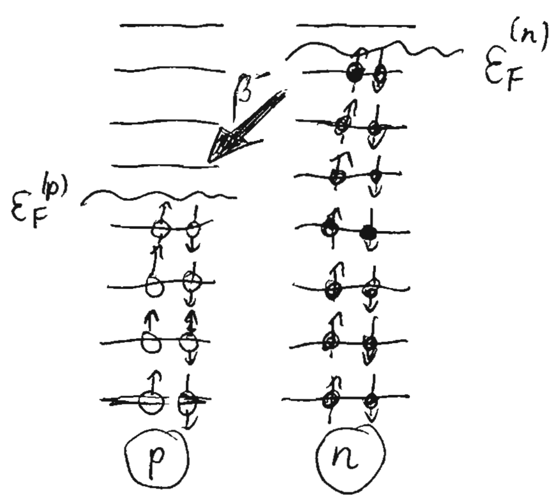
Again, for symmetric nuclear matter with $Z = N = \frac{A}{2}$ we obtain

$$\boxed{\varepsilon_F = \frac{\hbar^2 k_F^2}{2m_N} = \frac{(\hbar c)^2 k_F^2}{2 \cdot (m_N c^2)} \simeq 37\ \text{MeV}}$$

where we have used $\hbar c = 197.3$ MeV·fm and $m_N c^2 \simeq 939$ MeV.

Origin of symmetry energy

Finally, the free Fermi gas model gives a simple explanation for the symmetry energy. Assume, we have more neutrons than protons. Because of the <u>Pauli principle</u>, it is energetically



favorable to "convert" some of the neutrons into protons via <u>β^- decay:</u> $n \rightarrow p + e^- + \bar{\nu}_e$.

Everything else being equal, this lowers the total energy of the system. Similar considerations apply if we have more protons than neutrons.

The symmetry energy $E_{sym} \propto (N-Z)^2$ has a minimal value for $N = Z$ and shows exactly the behavior discussed here.

Further details, see K. Heyde textbook, chapter 8.